

THE YOKONUMA-HECKE ALGEBRAS AND THE HOMFLYPT POLYNOMIAL

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ABSTRACT. We compare the invariants for classical knots and links defined using the Juyumaya trace on the Yokonuma-Hecke algebras with the HOMFLYPT polynomial. We show that these invariants do not coincide with the HOMFLYPT except in a few trivial cases.

INTRODUCTION

The Yokonuma-Hecke algebras $Y_{d,n}(u)$ were introduced by Yokonuma [Yo] in the context of Chevalley groups, as generalizations of the Iwahori-Hecke algebras. The algebras $Y_{d,n}(u)$ may be also viewed as quotients of the framed braid group algebra over a quadratic relation (see (2.2)) involving the framing generators by means of certain weighted idempotents e_i . Thus the classical braid groups are also represented in the algebras $Y_{d,n}(u)$.

In [Ju] Juyumaya constructed a unique linear Markov trace tr on the algebras $Y_{d,n}(u)$, depending on d parameters, z, x_1, \dots, x_{d-1} . The trace tr was used subsequently in [JuLa2] for defining isotopy invariants for framed knots. As it turned out, the trace tr would not re-scale directly according to the braid equivalence moves. Therefore, certain conditions had to be imposed, implying that the trace parameters x_1, \dots, x_{d-1} had to satisfy a non linear system of equations, the so-called *E-system* (see (2.13)). Gérardin proved that the solutions of the E-system are parametrized by the non-empty subsets S of $\mathbb{Z}/d\mathbb{Z}$ (see Appendix of [JuLa2]). Given now any solution of the E-system, 2-variable isotopy invariants for framed, classical and singular knots were constructed respectively in [JuLa2, JuLa3, JuLa4].

For classical knots we have the well-known HOMFLYPT or 2-variable Jones polynomial P [Jo], which is determined by the Ocneanu trace (with parameter ζ) on the Iwahori-Hecke algebras $\mathcal{H}_n(q)$ of type A . Therefore, it is natural to ask how the invariant P compares with every invariant Δ_S derived from the Juyumaya trace on the algebras $Y_{d,n}(u)$, for any $d \in \mathbb{N}$ and for any non-empty subset S of $\mathbb{Z}/d\mathbb{Z}$. Computational data so far do not indicate that one invariant is topologically stronger than the other (see [CJKL]).

In order to compare the knot invariants P and Δ_S , we would like to be able to specialize the indeterminates x_1, \dots, x_{d-1} to a solution of the E-system as early as possible during the construction of Δ_S . This goal is achieved in Section 3 with the construction of the linear map φ on $Y_{d,n}(u)$. However, as we show, there is no appropriate algebra homomorphism between $Y_{d,n}(u)$ and $\mathcal{H}_n(q)$, unless $E := \text{tr}(e_i) = 1$, and this makes a connection between the corresponding trace functions impossible. In this paper we prove that the invariants P and Δ_S do not coincide except in a few trivial cases, that is, $u = 1$ or $q = 1$ or $E = 1$, given by Theorem 5. In fact we show (Theorem 6)

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that these are the only cases where the one invariant is a scalar multiple of the other, with scalars in $\mathbb{C}(q, \zeta, u, z, E)$.

The paper is organized as follows: In the first two sections we present some preliminary results on Iwahori-Hecke algebras and Yokonuma-Hecke algebras. In Section 3 we introduce the *specialized Juyumaya trace*, where the indeterminates x_1, \dots, x_{d-1} specialize to complex numbers, and we show that it factors through the linear map φ that we construct. We compare the specialized Juyumaya trace with the Ocneanu trace and we obtain one case (when $E = 1$) where the invariants P and Δ_S coincide. In the last two sections we proceed with comparing further the invariants, in order to obtain all cases where they coincide. More precisely, in Section 4 we give some necessary conditions for P and Δ_S to coincide, by evaluating the invariants on specific braid words. Finally, in Section 5 we prove, with the use of an elaborate induction, that these conditions are also sufficient.

1. THE 2-VARIABLE JONES OR HOMFLYPT POLYNOMIAL

1.1. *The symmetric group \mathfrak{S}_n .* The symmetric group \mathfrak{S}_n is generated by the transpositions s_1, s_2, \dots, s_{n-1} , with $s_i = (i, i+1)$, subject to the relations:

$$(1.1) \quad \begin{aligned} s_i s_j &= s_j s_i && \text{for } |i - j| > 1; \\ s_{i+1} s_i s_{i+1} &= s_i s_{i+1} s_i && \text{for all } i; \\ s_i^2 &= 1 && \text{for all } i. \end{aligned}$$

Let $S = \{s_1, s_2, \dots, s_{n-1}\}$ and let $w \in \mathfrak{S}_n$. Then $w = s_{i_1} s_{i_2} \dots s_{i_r}$, with $s_{i_j} \in S$, is an *expression* for w . If r is minimal such that there exists an expression $w = s_{i_1} s_{i_2} \dots s_{i_r}$, then this expression is called *reduced* and r is called the *length* of w . We denote the length of w by $\ell(w)$.

1.2. *Conjugacy classes of \mathfrak{S}_n .* The conjugacy classes of \mathfrak{S}_n correspond to the cycle types of permutations; that is, two elements of \mathfrak{S}_n are conjugate in \mathfrak{S}_n if and only if they consist of the same number of disjoint cycles of the same lengths. It is well-known that the conjugacy classes of \mathfrak{S}_n are naturally parametrized by the partitions μ of n . If μ has non-zero parts μ_1, μ_2, \dots , then we take $w_\mu := s_{i_k} \dots s_{i_2} s_{i_1}$ as representative in the class labelled by μ , where $\{i_1, i_2, \dots, i_k\}$ is the set obtained from $\{1, 2, \dots, n\}$ by removing the integers $\mu_1, \mu_1 + \mu_2, \mu_1 + \mu_2 + \mu_3, \dots$. For example, if $n = 8$ and $\mu = (4, 3, 1)$, then $\mu_1 = 4$, $\mu_1 + \mu_2 = 7$ and $\mu_1 + \mu_2 + \mu_3 = 8$, whence $w_\mu = s_6 s_5 s_3 s_2 s_1$. The point about choosing these representatives is that w_μ has minimal length in its conjugacy class, that is, we have $\ell(w_\mu) \leq \ell(w)$ for any $w \in \mathfrak{S}_n$ which is conjugate to w_μ . Now let

$$\mathfrak{D} := \{s_{i_k} \dots s_{i_2} s_{i_1} \mid i_1 < i_2 < \dots < i_k\} \subset \mathfrak{S}_n.$$

We obviously have $w_\mu \in \mathfrak{D}$ for every partition μ of n . Moreover, if $w = s_{i_k} \dots s_{i_2} s_{i_1} \in \mathfrak{D}$, one can easily check that, because of its cycle type, w has minimal length in its conjugacy class.¹

The following result, which relates elements of minimal length in a conjugacy class, will be useful in Subsection 3.3.

Theorem 1. [GePf, Theorem 3.2.9] *Let C be a conjugacy class of \mathfrak{S}_n and let w, w' be two elements of minimal length in C . Then w and w' are “strongly conjugate” in \mathfrak{S}_n , that is, there exists a finite sequence $w_0 = w, w_1, \dots, w_r = w'$ such that, for all $i = 0, 1, \dots, r-1$,*

$$\ell(w_i) = \ell(w_{i+1}), \quad w_{i+1} = x_i w_i x_i^{-1} \quad \text{and} \quad \ell(x_i w_i) = \ell(x_i) + \ell(w_i) \quad \text{or} \quad \ell(w_i x_i^{-1}) = \ell(w_i) + \ell(x_i^{-1})$$

for some elements $x_i \in \mathfrak{S}_n$.

¹This result also follows from [GePf, Lemma 3.1.14], since w is a Coxeter element of the parabolic subgroup W_J of \mathfrak{S}_n , where $J = \{s_{i_1}, s_{i_2}, \dots, s_{i_k}\} \subseteq S$.

1.3. *The Iwahori-Hecke algebra $\mathcal{H}_n(q)$.* Let $q \in \mathbb{C} \setminus \{0\}$. The Iwahori-Hecke algebra $\mathcal{H}_n(q)$ of type A is the \mathbb{C} -associative algebra with presentation on generators G_1, G_2, \dots, G_{n-1} , and relations:

$$(1.2) \quad \begin{array}{lll} (b_1) & G_i G_j & = G_j G_i \quad \text{for } |i - j| > 1; \\ (b_2) & G_{i+1} G_i G_{i+1} & = G_i G_{i+1} G_i \quad \text{for all } i; \\ (h) & G_i^2 & = (q - 1)G_i + q \quad \text{for all } i. \end{array}$$

The relations (b_1) and (b_2) are defining relations for the classical Artin braid group B_n ; hence $\mathcal{H}_n(q)$ can be viewed as the quotient of the group algebra $\mathbb{C}B_n$ over the quadratic relations (h) . Moreover, for $q = 1$ the algebra $\mathcal{H}_n(1)$ is isomorphic to the group algebra of the symmetric group $\mathbb{C}\mathfrak{S}_n$. If $w \in \mathfrak{S}_n$ and $w = s_{i_1} s_{i_2} \dots s_{i_r}$ is a reduced expression, we set $G_w := G_{i_1} G_{i_2} \dots G_{i_r}$. The set

$$\{G_w \mid w \in \mathfrak{S}_n\}$$

is the “standard” \mathbb{C} -basis of $\mathcal{H}_n(q)$. Now, the following set forms another linear \mathbb{C} -basis for $\mathcal{H}_n(q)$ [Jo, §4]:

$$\mathcal{S}_\mathcal{H} = \{(G_{i_1} \dots G_{i_1-r_1})(G_{i_2} \dots G_{i_2-r_2}) \dots (G_{i_p} \dots G_{i_p-r_p}) \mid 1 \leq i_1 < \dots < i_p \leq n-1\}.$$

Note that all generators G_i are invertible in $\mathcal{H}_n(q)$, with

$$(1.3) \quad G_i^{-1} = q^{-1}G_i + (q^{-1} - 1) \quad \text{for all } i.$$

1.4. *Computation formulas in the Iwahori-Hecke algebra.* Let $m \in \mathbb{N}$. It is easy to check that

$$G_i^m = (q^{m-1} - q^{m-2} + \dots + (-1)^{m-2}q + (-1)^{m-1})G_i + (q^{m-1} - q^{m-2} + \dots + (-1)^{m-2}q).$$

Hence, if m is even, we have

$$(1.4) \quad G_i^m = \left(\frac{q^m - 1}{q + 1} \right) G_i + \left(\frac{q^m - 1}{q + 1} + 1 \right),$$

and if m is odd, we have

$$(1.5) \quad G_i^m = \left(\frac{q^m + 1}{q + 1} \right) G_i + \left(\frac{q^m + 1}{q + 1} - 1 \right).$$

1.5. *The Ocneanu trace τ .* The natural inclusions $B_n \subset B_{n+1}$ of the classical braid groups give rise to the algebra inclusions:

$$\mathbb{C}B_0 \subset \mathbb{C}B_1 \subset \mathbb{C}B_2 \subset \dots$$

(setting $\mathbb{C}B_0 := \mathbb{C}$), which in turn induce the following algebra inclusions:

$$(1.6) \quad \mathcal{H}_0(q) \subset \mathcal{H}_1(q) \subset \mathcal{H}_2(q) \subset \dots$$

(setting $\mathcal{H}_0(q) := \mathbb{C}$). Then we have the following.

Theorem 2. [Jo, Theorem 5.1] *Let $\zeta \in \mathbb{C} \setminus \{0\}$ be an indeterminate. There exists a unique linear Markov trace*

$$\tau : \bigcup_{n \geq 0} \mathcal{H}_n(q) \longrightarrow \mathbb{C}[\zeta]$$

defined inductively on $\mathcal{H}_n(q)$ for all n , by the following rules:

$$\begin{array}{ll} \tau(hh') & = \tau(h'h) \\ \tau(1) & = 1 \\ \tau(hG_n) & = \zeta \tau(h) \end{array} \quad (\text{Markov property})$$

where $h, h' \in \mathcal{H}_n(q)$.

The trace τ is the *Ocneanu trace* with parameter ζ . Diagrammatically, in the second rule, 1 corresponds to the identity braid on any number of strands. The third rule is the so-called Markov property of the trace. One can look at the left-hand illustration of Figure 1 for a topological interpretation of the Markov property.

Another characterization of the Ocneanu trace can be given as follows. Every trace function is uniquely determined by its values on the basis elements G_w , where w runs over a certain set of representatives of the various conjugacy classes of \mathfrak{S}_n [GePf, 8.2.6]. Following [GePf, 3.1.16], these particular representatives are the elements w_μ , defined in §1.2 for every partition μ of n . Applying the defining formula for the Ocneanu trace τ to the element G_{w_μ} , we see that

$$(1.7) \quad \tau(G_{w_\mu}) = \zeta^{\ell(w_\mu)}.$$

Conversely, if ψ is any trace function on $\mathcal{H}_n(q)$ such that $\psi(G_{w_\mu}) = \zeta^{\ell(w_\mu)}$ for all partitions μ of n , then $\psi = \tau$.

1.6. The HOMFLYPT polynomial. Let now \mathcal{L} be the set of isotopy classes of oriented links in S^3 . We know from Jones' construction [Jo] that, in order to obtain a link invariant according to the Markov equivalence for braids, the closed braids $\widehat{\alpha}$, $\widehat{\alpha\sigma_n}$ and $\widehat{\alpha\sigma_n^{-1}}$ have to be assigned the same value for any braid $\alpha \in B_n$. Therefore, in order to obtain a link invariant from the trace τ , it has to be re-scaled, so that $\tau(hG_n^{-1}) = \tau(hG_n)$ for all $h \in \mathcal{H}_n(q)$, and also normalized, so that the closed braids $\widehat{\alpha}$ and $\widehat{\alpha\sigma_n}$ be assigned the same value. Set now

$$\lambda_{\mathcal{H}} := \frac{\zeta + (1 - q)}{q\zeta}.$$

Definition 1. [Jo, Definition 6.1] We define a map P on the set \mathcal{L} by defining P on the closure $\widehat{\alpha}$ of any braid $\alpha \in B_n$, for all $n \in \mathbb{N}$, as follows:

$$P(\widehat{\alpha}) := \left(-\frac{1 - \lambda_{\mathcal{H}}q}{\sqrt{\lambda_{\mathcal{H}}}(1 - q)} \right)^{n-1} (\sqrt{\lambda_{\mathcal{H}}})^{\epsilon(\alpha)} (\tau \circ \pi)(\alpha)$$

where $\pi : \mathbb{C}B_n \rightarrow \mathcal{H}_n(q)$ is the natural algebra epimorphism that maps the braid generator σ_i to the algebra generator G_i , and $\epsilon(\alpha)$ is the sum of the exponents of the braid generators in the braid word α . Equivalently, by setting

$$D_{\mathcal{H}} := -\frac{1 - \lambda_{\mathcal{H}}q}{\sqrt{\lambda_{\mathcal{H}}}(1 - q)} = \frac{1}{\zeta\sqrt{\lambda_{\mathcal{H}}}}$$

we can write:

$$P(\widehat{\alpha}) = (D_{\mathcal{H}})^{n-1} (\sqrt{\lambda_{\mathcal{H}}})^{\epsilon(\alpha)} (\tau \circ \pi)(\alpha).$$

As it turns out, the map P is well-defined on \mathcal{L} and it defines the well-known *2-variable Jones or HOMFLYPT polynomial*, an isotopy invariant of classical knots and links. This map depends on the quadratic relation (1.2)(h), so an automorphism of the Iwahori-Hecke algebra $\mathcal{H}_n(q)$ may give rise to a different map. However, one can easily check that the map P' induced by the automorphism

$$(1.8) \quad G_i \mapsto -q^{-1}G_i$$

is equal to P (if $G'_i := -q^{-1}G_i$, then $G_i'^2 = (q^{-1} - 1)G'_i + q^{-1}$ and $\tau(G'_i) = -q^{-1}\zeta$). We will need this later.

2. KNOT INVARIANTS FROM THE YOKONUMA-HECKE ALGEBRAS

2.1. *The Yokonuma-Hecke algebra $Y_{d,n}(u)$.* In the sequel we fix $d \in \mathbb{N}$. Let $u \in \mathbb{C} \setminus \{0\}$. The Yokonuma-Hecke algebra, denoted by $Y_{d,n}(u)$, is a \mathbb{C} -associative algebra generated by the elements

$$g_1, \dots, g_{n-1}, t_1, \dots, t_n$$

subject to the following relations:

$$(2.1) \quad \begin{array}{llll} (b_1) & g_i g_j & = & g_j g_i & \text{for } |i - j| > 1 \\ (b_2) & g_i g_j g_i & = & g_j g_i g_j & \text{for } |i - j| = 1 \\ (f_1) & t_i t_j & = & t_j t_i & \text{for all } i, j \\ (f_2) & t_j g_i & = & g_i t_{s_i(j)} & \text{for all } i, j \\ (f_3) & t_j^d & = & 1 & \text{for all } j \end{array}$$

where s_i is the transposition $(i, i + 1)$, together with the extra quadratic relations:

$$(2.2) \quad g_i^2 = 1 + (u - 1) e_i + (u - 1) e_i g_i \quad \text{for all } i$$

where

$$(2.3) \quad e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s}.$$

It is easily verified that the elements e_i are idempotents in $Y_{d,n}(u)$. Also, that the elements g_i are invertible, with

$$(2.4) \quad g_i^{-1} = g_i + (u^{-1} - 1) e_i + (u^{-1} - 1) e_i g_i.$$

The relations (b_1) , (b_2) , (f_1) and (f_2) are defining relations for the classical framed braid group $\mathcal{F}_n \cong \mathbb{Z}^n \rtimes B_n$, with the t_j 's being interpreted as the 'elementary framings' (framing 1 on the j th strand). The relations $t_j^d = 1$ mean that the framing of each braid strand is regarded modulo d . Thus, the algebra $Y_{d,n}(u)$ arises naturally as a quotient of the framed braid group algebra over the modular relations (f_3) and the quadratic relations (2.2) [Ju]. Moreover, relations (2.1) are defining relations for the modular framed braid group $\mathcal{F}_{d,n} \cong (\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n$, so the algebra $Y_{d,n}(u)$ can be also seen as a quotient of the modular framed braid group algebra over the quadratic relations (2.2).

From the above, the algebra $Y_{d,n}(u)$ has natural topological interpretation in the context of framed braids and framed knots. However, in [JuLa3] a different topological interpretation to $Y_{d,n}(u)$ was given, in relation to classical knots and links. Namely, viewing the t_j 's only as formal generators and ignoring their framing interpretation, we have by relations (b_1) and (b_2) that the classical braid group B_n is represented in $Y_{d,n}(u)$.

The Yokonuma-Hecke algebra was originally introduced by T. Yokonuma [Yo]. For $d = 1$, the algebra $Y_{1,n}(u)$ coincides with the Iwahori-Hecke algebra $\mathcal{H}_n(u)$ of type A . For more details and for further topological interpretations, see [JuLa1, JuLa2, JuLa3, JuLa4] and references therein.

2.2. *Computation formulas in the Yokonuma-Hecke algebra.* Let $i, k \in \{1, 2, \dots, n\}$ and set

$$(2.5) \quad e_{i,k} := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_k^{d-s}.$$

Clearly $e_{i,k} = e_{k,i}$ and it can be easily deduced that $e_{i,k}^2 = e_{i,k}$. Note that $e_{i,i} = 1$ and that $e_{i,i+1} = e_i$. Now, in $Y_{d,n}(u)$ the following relations hold (see [JuLa1, Lemma 4, Proposition 5]):

$$\begin{aligned}
(2.6) \quad & \begin{aligned} t_j e_i &= e_i t_j \\ e_j e_i &= e_i e_j \\ g_j e_i &= e_i g_j \end{aligned} & \text{for } j \neq i-1, i+1 \\
& \begin{aligned} g_{i-1} e_i &= e_{i-1, i+1} g_{i-1} \\ e_i g_{i-1} &= g_{i-1} e_{i-1, i+1} \\ g_{i+1} e_i &= e_{i, i+2} g_{i+1} \\ e_i g_{i+1} &= g_{i+1} e_{i, i+2} \end{aligned} \\
& e_j g_i g_j = g_i g_j e_i & \text{for } |i-j| = 1.
\end{aligned}$$

Note that, using (2.4), relations (2.6) are also valid if all g_k 's are replaced by their inverses g_k^{-1} . Moreover, the following relations hold in $Y_{d,n}(u)$.

Lemma 1. *Let $i, k \in \{1, 2, \dots, n\}$. We have*

$$t_i e_{i,k} = t_k e_{i,k}.$$

In particular,

$$t_i e_i = t_{i+1} e_i.$$

Proof. We have

$$t_i e_{i,k} = \frac{1}{d} \sum_{s=0}^{d-1} t_i^{s+1} t_k^{d-s} = \frac{1}{d} \sum_{r=1}^d t_i^r t_k^{d-r+1} = \frac{1}{d} \left(\sum_{r=1}^{d-1} t_i^r t_k^{d-r+1} + t_k \right) = t_k \left(\frac{1}{d} \sum_{r=0}^{d-1} t_i^r t_k^{d-r} \right) = t_k e_{i,k}.$$

□

The following equalities are easy to check (see, for example, [JuLa2, Lemma1]):

Lemma 2. *Let $m \in \mathbb{N}$. Then, if m is even, we have*

$$g_i^m = \frac{u^m - 1}{u + 1} e_i g_i + \frac{u^m - 1}{u + 1} e_i + 1,$$

and if m is odd, we have

$$g_i^m = \frac{u^m - u}{u + 1} e_i g_i + \frac{u^m - u}{u + 1} e_i + g_i.$$

2.3. The Juyumaya trace tr . The natural inclusions $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ of the classical framed braid groups induce natural inclusions $\mathcal{F}_{d,n} \subset \mathcal{F}_{d,n+1}$ of modular framed braid groups and these give rise to the algebra inclusions:

$$\mathbb{C}\mathcal{F}_{d,0} \subset \mathbb{C}\mathcal{F}_{d,1} \subset \mathbb{C}\mathcal{F}_{d,2} \subset \dots$$

(setting $\mathbb{C}\mathcal{F}_{d,0} := \mathbb{C}$), which in turn induce the following algebra inclusions:

$$(2.7) \quad Y_{d,0}(u) \subset Y_{d,1}(u) \subset Y_{d,2}(u) \subset \dots$$

(setting $Y_{d,0}(u) := \mathbb{C}$). Then we have the following:

Theorem 3. [Ju, Theorem 12] *Let $z, x_1, \dots, x_{d-1} \in \mathbb{C} \setminus \{0\}$ be indeterminates. There exists a unique linear Markov trace*

$$\text{tr} : \bigcup_{n \geq 0} Y_{d,n}(u) \longrightarrow \mathbb{C}[z, x_1, \dots, x_{d-1}]$$

defined inductively on $Y_{d,n}(u)$ for all n , by the following rules:

$$\begin{aligned} \text{tr}(ab) &= \text{tr}(ba) \\ \text{tr}(1) &= 1 \\ \text{tr}(ag_n) &= z \text{tr}(a) && (\text{Markov property}) \\ \text{tr}(at_{n+1}^m) &= x_m \text{tr}(a) && (m = 1, \dots, d-1) \end{aligned}$$

where $a, b \in Y_{d,n}(u)$.

We shall call the trace tr the *Juyumaya trace* with parameters z, x_1, \dots, x_{d-1} . Diagrammatically, in the second rule, 1 corresponds to the identity braid on any number of strands with all framings zero. The following figure gives the topological interpretations of the last two rules.

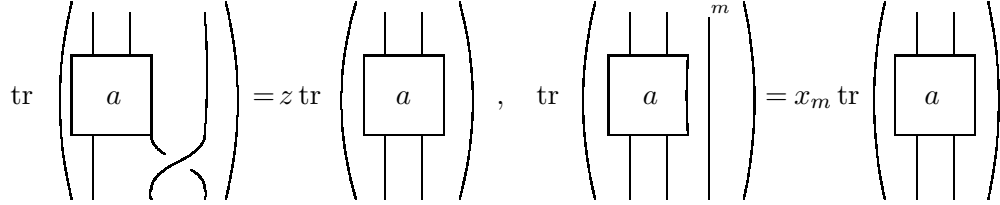


FIGURE 1. Topological interpretations of the trace rules

The trace rules yield the following relations for all i :

$$(2.8) \quad \text{tr}(e_i) = \frac{1}{d} \sum_{s=0}^{d-1} x_s x_{d-s} =: E \quad \text{and} \quad \text{tr}(e_i g_i) = \text{tr}(g_i) = z.$$

Using (2.8), Lemma 2 implies that the following relations hold: For $m \in \mathbb{Z}^{>0}$, we have

$$(2.9) \quad \text{tr}(g_i^m) = \left(\frac{u^m - 1}{u + 1} \right) z + \left(\frac{u^m - 1}{u + 1} \right) E + 1 \quad \text{if } m \text{ is even}$$

and

$$(2.10) \quad \text{tr}(g_i^m) = \left(\frac{u^m + 1}{u + 1} \right) z + \left(\frac{u^m + 1}{u + 1} \right) E - E \quad \text{if } m \text{ is odd.}$$

2.4. An inductive basis for the Yokonuma-Hecke algebra. The key in the construction of the trace tr is that $Y_{d,n}(u)$ has a ‘nice’ inductive linear \mathbb{C} -basis. Namely, every element of $Y_{d,n+1}(u)$ is a unique linear combination of words of the following types:

$$(2.11) \quad w_n g_n g_{n-1} \dots g_i t_i^k \quad \text{or} \quad w_n t_{n+1}^k \quad (k \in \mathbb{Z}/d\mathbb{Z})$$

where $w_n \in Y_{d,n}(u)$. Thus, the above words furnish an inductive basis for $Y_{d,n+1}(u)$, every element of which involves g_n or a power of t_{n+1} at most once.

2.5. The split property for the Yokonuma-Hecke algebra. Due to the relations (2.1)(f₁) and (2.1)(f₂), every monomial w in $Y_{d,n}(u)$ can be written in the form

$$w = t_1^{k_1} \dots t_n^{k_n} \cdot \sigma$$

where $k_1, \dots, k_n \in \mathbb{Z}/d\mathbb{Z}$ and σ is a word in g_1, \dots, g_{n-1} . That is, w splits into the ‘framing part’ $t_1^{k_1} \dots t_n^{k_n}$ and the ‘braiding part’ σ . Applying further the braid relations (2.1)(b₁) and (2.1)(b₂) and the quadratic relations (2.2), we deduce that the following set is a \mathbb{C} -basis for $Y_{d,n}(u)$ [Ju, JuLa1]:

$$\mathcal{S}_Y = \left\{ t_1^{k_1} \dots t_n^{k_n} (g_{i_1} \dots g_{i_1-r_1}) (g_{i_2} \dots g_{i_2-r_2}) \dots (g_{i_p} \dots g_{i_p-r_p}) \mid \begin{array}{l} k_1, \dots, k_n \in \mathbb{Z}/d\mathbb{Z} \\ 1 \leq i_1 < \dots < i_p \leq n-1 \end{array} \right\}$$

2.6. *The E-system.* Let now \mathcal{L} be, as above, the set of isotopy classes of oriented links in S^3 . As mentioned in Subsection 1.6, likewise here, in order to obtain a link invariant from the trace tr according to the Markov equivalence for braids, the trace has to be: normalized, so that the closed braids $\widehat{\alpha}$ and $\widehat{\alpha\sigma_n}$ ($\alpha \in B_n$) be assigned the same value of the invariant, and re-scaled, so that the closed braids $\widehat{\alpha\sigma_n^{-1}}$ and $\widehat{\alpha\sigma_n}$ ($\alpha \in B_n$) be assigned the same value of the invariant.

Trying to do that, it was shown in [JuLa2] that tr does not re-scale directly (being the only known Markov trace with this property). Indeed, for $\alpha \in Y_{d,n}(u)$, we compute:

$$\text{tr}(\alpha g_n^{-1}) = \text{tr}(\alpha g_n) + (u^{-1} - 1)\text{tr}(\alpha e_n) + (u^{-1} - 1)\text{tr}(\alpha e_n g_n).$$

Now, although

$$(2.12) \quad \text{tr}(\alpha e_n g_n) = \text{tr}(\alpha g_n) = z \text{tr}(\alpha) = \text{tr}(g_n) \text{tr}(\alpha)$$

we have that $\text{tr}(\alpha e_n)$ does not factor through $\text{tr}(\alpha)$, that is,

$$\text{tr}(\alpha e_n) \neq \text{tr}(e_n) \text{tr}(\alpha)$$

leading to the fact that $\text{tr}(\alpha g_n^{-1})$ does not factor through $\text{tr}(\alpha)$, that is,

$$\text{tr}(\alpha g_n^{-1}) \neq \text{tr}(g_n^{-1}) \text{tr}(\alpha).$$

Forcing $\text{tr}(\alpha e_n) = \text{tr}(e_n) \text{tr}(\alpha)$ yields that the trace parameters x_1, \dots, x_{d-1} have to satisfy the so-called *E-system*. The E-system is a non-linear system of equations of the form:

$$(2.13) \quad \sum_{s=0}^{d-1} x_{m+s} x_{d-s} = x_m \sum_{s=0}^{d-1} x_s x_{d-s} \quad (1 \leq m \leq d-1)$$

where the sub-indices on the x_j 's are regarded modulo d and $x_0 := 1$. Equivalently, the E-system is written as

$$E^{(m)} = x_m E$$

where

$$E^{(m)} := \frac{1}{d} \sum_{s=0}^{d-1} x_{m+s} x_{d-s} \quad \text{and} \quad E := E^{(0)} = \frac{1}{d} \sum_{s=0}^{d-1} x_s x_{d-s} = \text{tr}(e_i).$$

As it is shown in [JuLa2] (in the Appendix by Paul Gérardin), the solutions of the E-system are parametrized by the non-empty subsets of $\mathbb{Z}/d\mathbb{Z}$. Let $X_S = \{x_1, \dots, x_{d-1}\}$ be a solution of the E-system parametrized by the non-empty subset S of $\mathbb{Z}/d\mathbb{Z}$. Then, as it turns out [JuLa3],

$$(2.14) \quad E = \text{tr}(e_i) = \frac{1}{|S|} \quad (1 \leq i \leq n-1).$$

Moreover, if X_S satisfies the E-condition, then we have [JuLa2, Theorem 7]:

$$(2.15) \quad \text{tr}(\alpha e_n) \stackrel{\text{E}}{=} \text{tr}(\alpha) \text{tr}(e_n) = E \text{tr}(\alpha) \quad (\alpha \in Y_{d,n}(u)).$$

Notation. The symbol ' $\stackrel{\text{E}}{=}$ ' will stand for '=' up to the E-condition, that is, with a given solution of the E-system.

2.7. *The Case $E = 1$.* The ‘trivial’ solutions of the E-system are the ones parametrized by the singleton subsets of $\mathbb{Z}/d\mathbb{Z}$. By (2.14), if S is a singleton then $E = \text{tr}(e_i) = 1$. In this case, Gérardin has shown, in the Appendix of [JuLa2], that x_1 is a d -th root of unity and $x_m = x_1^m$ ($1 \leq m \leq d-1$). Consequently,

$$(2.16) \quad x_{k+l} = x_1^{k+l} = x_1^k x_1^l = x_k x_l \quad (k, l \in \mathbb{Z}/d\mathbb{Z}).$$

These solutions are not very interesting topologically, but we prove here that they have the following interesting property (a stronger version of (2.15)):

Proposition 1. *Let X_S be a solution of the E-system such that $E = 1$. Then*

$$\text{tr}(\beta e_j) \stackrel{E}{=} \text{tr}(\beta) \text{tr}(e_j) = \text{tr}(\beta) \quad (\beta \in Y_{d,n+1}(u), 1 \leq j \leq n).$$

Proof. It is enough to show that the above equality holds for the elements of the inductive basis given in (2.11). We will proceed by induction on n . Let $n = 1$ and let $k, l \in \mathbb{Z}/d\mathbb{Z}$. We have

$$\text{tr}(t_1^k g_1 t_1^l e_1) \stackrel{(2.6)}{=} \text{tr}(t_1^k g_1 e_1 t_1^l) \stackrel{(2.12)}{=} \text{tr}(t_1^k g_1 t_1^l).$$

Moreover, following Lemma 1, we obtain:

$$\text{tr}(t_1^k t_2^l e_1) = \text{tr}(t_1^{k+l} e_1) \stackrel{E}{=} \text{tr}(t_1^{k+l}) \text{tr}(e_1) = x_{k+l} \stackrel{(2.16)}{=} x_k x_l = \text{tr}(t_1^k t_2^l).$$

Now let $n > 1$ and assume that the statement of the proposition holds for smaller values of n .

Let $\beta = w_n g_n g_{n-1} \dots g_i t_i^k$ for some $k \in \mathbb{Z}/d\mathbb{Z}$ and some $w_n \in Y_{d,n}(u)$. Assume first that $j < n$. Then

$$\text{tr}(w_n g_n g_{n-1} \dots g_i t_i^k e_j) = z \text{tr}(w_n g_{n-1} \dots g_i t_i^k e_j).$$

By the induction hypothesis, the latter is equal to

$$z \text{tr}(w_n g_{n-1} \dots g_i t_i^k) = \text{tr}(w_n g_n g_{n-1} \dots g_i t_i^k),$$

so we are done. Now take $j = n$. If $i = n$, then, by (2.6) and (2.12):

$$\text{tr}(w_n g_n t_n^k e_n) = \text{tr}(w_n g_n t_n^k).$$

If $i < n$, then by (2.6):

$$\text{tr}(w_n g_n g_{n-1} \dots g_i t_i^k e_n) \stackrel{(2.1)(f_2)}{=} \text{tr}(w_n e_{n-1} g_n g_{n-1} \dots g_i t_i^k) = z \text{tr}(g_{n-1} \dots g_i t_i^k w_n e_{n-1}).$$

By the induction hypothesis, the latter is equal to $z \text{tr}(g_{n-1} \dots g_i t_i^k w_n)$, whence we deduce that

$$\text{tr}(w_n g_n g_{n-1} \dots g_i t_i^k e_n) = \text{tr}(w_n g_n g_{n-1} \dots g_i t_i^k).$$

Now let $\beta = w_n t_{n+1}^k$ for some $k \in \mathbb{Z}/d\mathbb{Z}$ and some $w_n \in Y_{d,n}(u)$. Assume again first that $j < n$. Applying the trace definition and the induction hypothesis we obtain:

$$\text{tr}(w_n t_{n+1}^k e_j) = x_k \text{tr}(w_n e_j) \stackrel{E}{=} x_k \text{tr}(w_n) = \text{tr}(w_n t_{n+1}^k).$$

Now take $j = n$. With the use of Lemma 1 and (2.15), we obtain:

$$\text{tr}(w_n t_{n+1}^k e_n) = \text{tr}(w_n t_n^k e_n) \stackrel{E}{=} \text{tr}(w_n t_n^k) \text{tr}(e_n) = \text{tr}(w_n t_n^k).$$

We need to show that, under the assumptions of the proposition:

$$\text{tr}(w_n t_n^k) = \text{tr}(w_n t_{n+1}^k) = x_k \text{tr}(w_n).$$

Again, it is enough to show this for the elements of the inductive basis of $Y_{d,n}(u)$. We have already shown it for $n = 1$ and we will proceed by induction on n . Let $w_n = w_{n-1} g_{n-1} g_{n-2} \dots g_i t_i^l$ for some $l \in \mathbb{Z}/d\mathbb{Z}$ and some $w_{n-1} \in Y_{d,n-1}(u)$. Then

$$\text{tr}(w_n t_n^k) = \text{tr}(w_{n-1} g_{n-1} g_{n-2} \dots g_i t_i^l t_n^k) = \text{tr}(w_{n-1} t_{n-1}^k g_{n-1} g_{n-2} \dots g_i t_i^l) = z \text{tr}(w_{n-1} t_{n-1}^k g_{n-2} \dots g_i t_i^l).$$

By the induction hypothesis, the latter is equal to

$$z \mathbf{x}_k \operatorname{tr}(w_{n-1} g_{n-2} \dots g_i t_i^l) = \mathbf{x}_k \operatorname{tr}(w_{n-1} g_{n-1} g_{n-2} \dots g_i t_i^l) = \mathbf{x}_k \operatorname{tr}(w_n).$$

Finally, let $w_n = w_{n-1} t_n^l$ for some $l \in \mathbb{Z}/d\mathbb{Z}$ and some $w_{n-1} \in Y_{d,n-1}(u)$. Then

$$\operatorname{tr}(w_n t_n^k) = \operatorname{tr}(w_{n-1} t_n^l t_n^k) = \operatorname{tr}(w_{n-1} t_n^{l+k}) = \mathbf{x}_{k+l} \operatorname{tr}(w_{n-1}) \stackrel{(2.16)}{=} \mathbf{x}_k \mathbf{x}_l \operatorname{tr}(w_{n-1}) = \mathbf{x}_k \operatorname{tr}(w_{n-1} t_n^l) = \mathbf{x}_k \operatorname{tr}(w_n).$$

□

Note that there are also non-trivial solutions of the E-system, with more interesting topological interpretations. For a detailed analysis see [JuLa2].

2.8. Invariants for classical knots from tr . Given a solution $X_S = \{\mathbf{x}_1, \dots, \mathbf{x}_{d-1}\}$ of the E-system, we wish to define a link isotopy invariant Δ_S . Let $\alpha \in B_n$. The E-condition guarantees that $\Delta_S(\widehat{\alpha\sigma_n}) = \Delta_S(\widehat{\alpha\sigma_n^{-1}})$. In order for $\Delta_S(\widehat{\alpha\sigma_n}) = \Delta_S(\widehat{\alpha})$ to hold, we need to normalize. Thus, by setting:

$$\lambda_Y := \frac{z + (1-u)E}{uz}$$

we define the following map on the set \mathcal{L} of oriented classical link types.

Definition 2. [JuLa3, Definition 3] For a solution X_S of the E-system parametrized by the subset S of $\mathbb{Z}/d\mathbb{Z}$, we define a map Δ_S on the set \mathcal{L} by defining Δ_S on the closure $\widehat{\alpha}$ of any braid $\alpha \in B_n$, for all $n \in \mathbb{N}$, as follows:

$$\Delta_S(\widehat{\alpha}) := \left(-\frac{1 - \lambda_Y u}{\sqrt{\lambda_Y}(1-u)E} \right)^{n-1} (\sqrt{\lambda_Y})^{\epsilon(\alpha)} (\operatorname{tr} \circ \delta)(\alpha)$$

where $\delta : \mathbb{C}B_n \rightarrow Y_{d,n}(u)$ is the natural algebra homomorphism that maps the braid generator σ_i to the algebra generator g_i , and $\epsilon(\alpha)$ is the sum of the exponents of the braid generators in the braid word α . Equivalently, by setting

$$D_Y := \frac{1 - \lambda_Y u}{\sqrt{\lambda_Y}(1-u)E} = \frac{1}{z\sqrt{\lambda_Y}}$$

we can write:

$$\Delta_S(\widehat{\alpha}) = D_Y^{n-1} (\sqrt{\lambda_Y})^{\epsilon(\alpha)} (\operatorname{tr} \circ \delta)(\alpha).$$

In [JuLa3] the following result was obtained:

Theorem 4. *For a solution X_S of the E-system, Δ_S is well-defined on the set \mathcal{L} , that is, it is a 2-variable isotopy invariant for oriented classical links, depending on the variables u, z .*

Note that for every $d \in \mathbb{N}$, the above construction provides us with $2^d - 1$ isotopy invariants for knots.

As shown in [CJKL], the invariants Δ_S (but not the trace tr) have the multiplicative property on connected sums of links. Further, it was shown in [JuLa3] that Δ_S satisfies a ‘closed’ cubic relation (closed in the sense of only involving braiding generators), which factors to the quadratic relation of the Iwahori–Hecke algebra $\mathcal{H}_n(u)$. Finally, it was shown in [JuLa2] that in the context of framed links the invariant Δ_S satisfies a skein relation. However, when Δ_S is seen as invariant of classical knots, this skein relation has no topological interpretation. This makes it very difficult to compare the invariants Δ_S with the HOMFLYPT polynomial using diagrammatic methods.

3. THE OCNEANU TRACE VS THE SPECIALIZED JUYUMAYA TRACE

In order to compare the knot invariants P and Δ_S , in view of the E-condition, we would like to be able to specialize the indeterminates x_1, \dots, x_{d-1} to a solution of the E-system as early as possible during the construction.

3.1. *The algebra homomorphism approach.* Our first approach to the above problem is to construct an algebra homomorphism $f : \bigcup_{n \geq 0} Y_{d,n}(u) \longrightarrow \bigcup_{n \geq 0} Y_{d,n}(u)$ such that

$$f(t_i^m) = x_m \quad (1 \leq m \leq d-1),$$

where $x_1, x_2, \dots, x_{d-1} \in \mathbb{C} \setminus \{0\}$. Since f is an algebra homomorphism, we must have

$$f(t_i^k t_i^l) = f(t_i^{k+l}) = x_{k+l} = x_k x_l = f(t_i^k) f(t_i^l) \quad \text{for } k, l \in \mathbb{Z}/d\mathbb{Z},$$

that is, x_1 is a d -th root of unity and $x_m = x_1^m$ ($1 \leq m \leq d-1$). In this case, the complex numbers x_1, x_2, \dots, x_{d-1} are a solution of the E-system such that $E = 1$. Moreover, we must have

$$f(g_i^2) = f(1 + (u-1)e_i + (u-1)e_i g_i) = 1 + (u-1) + (u-1)f(g_i) = u + (u-1)f(g_i) = f(g_i)^2,$$

whence we deduce that

$$f(g_i) = u \quad \text{or} \quad f(g_i) = -1.$$

All these restrictions that f imposes ($E = 1$, $f(g_i) \in \{u, -1\}$) make f an uninteresting mapping for our purposes (although it explains Cases 9 – 12 of Theorem 5 that we will see later). This is why in this section we will proceed with a step-by-step specialization $t_i^m \mapsto x_m$, namely with the construction of a ‘specialized’ Juyumaya trace, and a linear map φ on the Yokonuma-Hecke algebra through which the trace factors. This will allow us to conclude, in Subsection 3.4, that the invariants P and Δ_S coincide when $E = 1$.

3.2. *The specialized Juyumaya trace.*

Definition 3. Let $x_1, x_2, \dots, x_{d-1} \in \mathbb{C} \setminus \{0\}$ and consider the ring homomorphism

$$\begin{aligned} \theta : \mathbb{C}[z, x_1, \dots, x_{d-1}] &\longrightarrow \mathbb{C}[z] \\ z &\mapsto z \\ x_m &\mapsto x_m \quad (1 \leq m \leq d-1) \end{aligned}$$

The map θ shall be called the *specialization map*. We will call the composition

$$\theta \circ \text{tr} : \bigcup_{n \geq 0} Y_{d,n}(u) \longrightarrow \mathbb{C}[z]$$

the *specialized Juyumaya trace* with parameter z .

Note that in the case $d = 1$, when the algebra $Y_{1,n}(u)$ coincides with the Iwahori-Hecke algebra $\mathcal{H}_n(u)$, θ is simply the identity map on $\mathbb{C}[z]$ and the specialized Juyumaya trace $\theta \circ \text{tr} = \text{tr}$ coincides with the Ocneanu trace.

Remark 1. For a fixed θ , the specialized Juyumaya trace with parameter z is equal to the Juyumaya trace with parameters z, x_1, \dots, x_{d-1} . In this section, we will keep the notation $\theta \circ \text{tr}$ to avoid confusion. Hence, for a solution X_S of the E-system, the invariant Δ_S can be rewritten using the map θ as follows:

$$\Delta_S(\hat{\alpha}) = D_Y^{n-1} (\sqrt{\lambda_Y})^{\epsilon(\alpha)} (\theta \circ \text{tr} \circ \delta)(\alpha).$$

3.3. *Similarities with the Ocneanu trace.* In this subsection we will give another characterization of the specialized Juyumaya trace as follows. Let $w \in \mathfrak{S}_n$ and let $w = s_{i_1} s_{i_2} \dots s_{i_r}$ be a reduced expression. Then we can set $g_w := g_{i_1} g_{i_2} \dots g_{i_r}$. If $w, w' \in \mathfrak{S}_n$ are such that $\ell(w w') = \ell(w) + \ell(w')$, then we have

$$(3.1) \quad g_w g_{w'} = g_{w w'}.$$

Let μ be a partition of n and let w_μ be the corresponding element of \mathfrak{S}_n defined in §1.2. Applying the defining formula for the Juyumaya trace tr to the element g_{w_μ} , we see that $\text{tr}(g_{w_\mu}) = z^{\ell(w_\mu)}$, whence we deduce:

$$(3.2) \quad (\theta \circ \text{tr})(g_{w_\mu}) = z^{\ell(w_\mu)}.$$

We will show that, as in the case of Iwahori-Hecke algebra of type A , the specialized Juyumaya trace on $Y_{d,n}(u)$ is uniquely determined by its values on the elements g_{w_μ} , where μ runs over the partitions of n . That is, if ψ is any trace function on $Y_{d,n}(u)$ such that $\psi(g_{w_\mu}) = z^{\ell(w_\mu)}$ for all partitions μ of n , then $\theta \circ \psi = \theta \circ \text{tr}$. To achieve our aim, we shall first construct a linear map $\varphi : \bigcup_{n \geq 0} Y_{d,n}(u) \longrightarrow \bigcup_{n \geq 0} Y_{d,n}(u)$ with the property: $\text{tr} \circ \varphi = \theta \circ \text{tr}$.

Proposition 2. *Let θ be as above and set $x_0 := 1$. Let $\varphi : \bigcup_{n \geq 0} Y_{d,n}(u) \longrightarrow \bigcup_{n \geq 0} Y_{d,n}(u)$ be the linear map defined inductively on $Y_{d,n}(u)$, for all $n \in \mathbb{N}$, by the following rules:*

$$\begin{aligned} \varphi(1) &= 1 \\ \varphi(w_n g_n g_{n-1} \dots g_i t_i^k) &= g_n \varphi(w_n g_{n-1} \dots g_i t_i^k) \\ \varphi(w_n t_{n+1}^k) &= x_k \varphi(w_n) \end{aligned}$$

where $w_n \in Y_{d,n}(u)$ and $k \in \mathbb{Z}/d\mathbb{Z}$. Then we have:

$$(3.3) \quad \text{tr} \circ \varphi = \theta \circ \text{tr}.$$

Proof. It is enough to show that (3.3) holds on the elements of the inductive basis of $Y_{d,n+1}(u)$, and we will do this by induction on n . From now on, k and l are elements of $\mathbb{Z}/d\mathbb{Z}$.

First, let $n = 1$. We have

$$\text{tr} \left(\varphi(t_1^k g_1) \right) = \text{tr} \left(g_1 \varphi(t_1^k) \right) = \text{tr}(x_k g_1) = x_k z = \theta(x_k z) = \theta \left(\text{tr}(t_1^k g_1) \right)$$

and

$$\text{tr} \left(\varphi(t_1^l t_2^k) \right) = \text{tr} \left(x_k \varphi(t_1^l) \right) = \text{tr}(x_k x_l) = x_k x_l = \theta(x_k x_l) = \theta \left(\text{tr}(t_1^l t_2^k) \right).$$

Now assume that (3.3) holds for smaller values of n . We have

$$\text{tr} \left(\varphi(w_n g_n g_{n-1} \dots g_i t_i^k) \right) = \text{tr} \left(g_n \varphi(w_n g_{n-1} \dots g_i t_i^k) \right) = z \text{tr} \left(\varphi(w_n g_{n-1} \dots g_i t_i^k) \right),$$

since $\varphi(w_n g_{n-1} \dots g_i t_i^k) \in Y_{d,n}(u)$. By the induction hypothesis, the last term is equal to

$$z \theta \left(\text{tr}(w_n g_{n-1} \dots g_i t_i^k) \right) = \theta \left(z \text{tr}(w_n g_{n-1} \dots g_i t_i^k) \right) = \theta \left(\text{tr}(w_n g_n g_{n-1} \dots g_i t_i^k) \right).$$

Finally, we have

$$\text{tr} \left(\varphi(w_n t_{n+1}^k) \right) = \text{tr} (x_k \varphi(w_n)) = x_k \text{tr} (\varphi(w_n)).$$

By the induction hypothesis, the last term is equal to

$$x_k \theta \left(\text{tr}(w_n) \right) = \theta (x_k \text{tr}(w_n)) = \theta (\text{tr} (w_n t_{n+1}^k)).$$

□

Remark 2. The map φ is an intermediate construction between the algebra and the trace map. An analogue of the map φ can be constructed on the Iwahori-Hecke algebra $\mathcal{H}_n(q)$.

Remark 3. By virtue of Proposition 2, for a solution $X_S = \{x_1, \dots, x_{d-1}\}$ of the E-system, the invariant Δ_S is rewritten as:

$$\Delta_S(\hat{\alpha}) = D_Y^{n-1} (\sqrt{\lambda_Y})^{\epsilon(\alpha)} (\text{tr} \circ \varphi \circ \delta) (\alpha).$$

Moreover, in view of the discussion in Subsection 3.1, and since $\varphi(g_i) = g_i$ and $\varphi(t_i^m) = x_m$, the map φ of Proposition 2 provides us with the earliest possible specialization of x_1, \dots, x_{d-1} to X_S during the construction of Δ_S .

Proposition 2 implies that the specialized Juyumaya trace is uniquely determined by its values on the elements of the image of φ . We will now show that $\varphi(Y_{d,n}(u))$ is the subspace W_n of $Y_{d,n}(u)$ spanned by the elements $\{g_w\}_{w \in \mathfrak{D}}$, where

$$\mathfrak{D} = \{s_{i_k} \dots s_{i_2} s_{i_1} \mid i_1 < i_2 < \dots < i_k\} \subset \mathfrak{S}_n.$$

Proposition 3. *Let $n \in \mathbb{N}$ and let W_n be the \mathbb{C} -linear subspace of $Y_{d,n}(u)$ spanned by the elements $\{g_w\}_{w \in \mathfrak{D}}$. Then $\varphi(Y_{d,n}(u)) = W_n$.*

Proof. First note that we have $W_n \subset W_{n+1}$ and $g_n W_n \subset W_{n+1}$. Note also that $\varphi(1) = 1 \in W_n$.

We will first show that $\varphi(Y_{d,n}(u)) \subseteq W_n$. We will proceed by induction on n . Let $n = 1$. Then $\varphi(t_1^k) = x_k \cdot 1 \in W_1$, for all $k \in \mathbb{Z}/d\mathbb{Z}$. Now assume that $\varphi(Y_{d,n}(u)) \subseteq W_n$. In order to show that $\varphi(Y_{d,n+1}(u)) \subseteq W_{n+1}$, it is enough to show that the images of the elements of the inductive basis of $Y_{d,n+1}(u)$ under φ are contained in W_{n+1} . Let $w_n \in Y_{d,n}(u)$ and $k \in \mathbb{Z}/d\mathbb{Z}$. We have

$$\varphi(w_n g_n g_{n-1} \dots g_i t_i^k) = g_n \varphi(w_n g_{n-1} \dots g_i t_i^k) \in g_n W_n \subset W_{n+1}$$

and

$$\varphi(w_n t_{n+1}^k) = x_k \varphi(w_n) \in W_n \subset W_{n+1}.$$

We conclude that $\varphi(Y_{d,n+1}(u)) \subseteq W_{n+1}$.

On the other hand, let $w \in \mathfrak{D}$. Then $g_w = \varphi(g_w)$, and so $W_n \subseteq \varphi(Y_{d,n}(u))$. \square

Now let $w \in \mathfrak{D}$. Following the discussion in §1.2, w has minimal length in its conjugacy class in \mathfrak{S}_n . Suppose that the conjugacy class of w is parametrized by the partition μ of n . By Theorem 1, the elements w and w_μ are strongly conjugate, that is, there exists a finite sequence $w = w_0, w_1, \dots, w_r = w_\mu$ such that, for all $i = 0, 1, \dots, r-1$,

$$\ell(w_i) = \ell(w_{i+1}), \quad w_{i+1} = x_i w_i x_i^{-1} \quad \text{and} \quad \ell(x_i w_i) = \ell(x_i) + \ell(w_i) \quad \text{or} \quad \ell(w_i x_i^{-1}) = \ell(w_i) + \ell(x_i^{-1})$$

for some elements $x_i \in \mathfrak{S}_n$. Following (3.1) we deduce that, for all $i = 0, 1, \dots, r-1$, we have

$$g_{w_{i+1}} = g_{x_i} g_{w_i} g_{x_i}^{-1},$$

and thus,

$$\text{tr}(g_{w_{i+1}}) = \text{tr}(g_{w_i}).$$

In particular, we have

$$\text{tr}(g_w) = \text{tr}(g_{w_\mu}).$$

More generally, if ψ is any trace function on $Y_{d,n}(u)$, then we have

$$\psi(g_w) = \psi(g_{w_\mu}).$$

We conclude that the specialized Juyumaya trace on $Y_{d,n}(u)$ is uniquely determined by its values on the elements g_{w_μ} , where μ runs over the partitions of n .

3.4. Consequences on the case $E = 1$. Suppose now that the complex numbers x_1, x_2, \dots, x_{d-1} are solutions of the E-system such that $E = 1$, that is, x_1 is a d -th root of unity and $x_m = x_1^m$ ($1 \leq m \leq d-1$). In this case, we can define the algebra epimorphism

$$\begin{aligned} \gamma : Y_{d,n}(u) &\longrightarrow \mathcal{H}_n(u) \\ g_i &\mapsto G_i \\ t_i^m &\mapsto x_m \quad (1 \leq m \leq d-1). \end{aligned}$$

This is indeed an algebra homomorphism, since it respects all the defining relations of the algebra $Y_{d,n}(u)$. In particular:

$$\gamma(g_i^2) = \gamma(1 + (u-1)e_i + (u-1)e_i g_i) = u + (u-1)G_i = G_i^2 = \gamma(g_i)^2$$

and

$$\gamma(t_i^k t_i^l) = \gamma(t_i^{k+l}) = x_{k+l} \stackrel{E}{=} x_k x_l = \gamma(t_i^k) \gamma(t_i^l) \quad \text{for } k, l \in \mathbb{Z}/d\mathbb{Z}.$$

The map γ is also clearly surjective.

Remark 4. Note that the map γ is not an algebra homomorphism if $E \neq 1$, because neither of the above equalities holds.

Now consider the Ocneanu trace on $\mathcal{H}_n(u)$ with parameter $\zeta = z$. The composition $\tau \circ \gamma$ is a Markov trace on $Y_{d,n}(u)$ which takes the same values as the specialized Juyumaya trace on the elements g_{w_μ} , where μ runs over the partitions of n . We deduce that

$$(3.4) \quad \tau \circ \gamma = \theta \circ \text{tr}.$$

The following result is a consequence of (3.4).

Proposition 4. *Let X_S be a solution of the E-system such that $E = 1$. Let tr be the Juyumaya trace on $Y_{d,n}(u)$ with parameters z, X_S , and let τ be the Ocneanu trace on $\mathcal{H}_n(q)$ with parameter ζ . If we take $u = q$ and $z = \zeta$, then*

$$(\tau \circ \pi)(\alpha) = (\text{tr} \circ \delta)(\alpha) \quad (\alpha \in B_n)$$

for all $n \in \mathbb{N}$.

Proof. Let $\alpha \in B_n$. By definition of the map γ , we have $(\gamma \circ \delta)(\alpha) = \pi(\alpha)$. Following Remark 1, the Juyumaya trace tr with parameters z, X_S is equal to the corresponding specialized Juyumaya trace. Now Equation 3.4 yields the desired result. \square

Under the assumptions of Proposition 4, we automatically obtain $\lambda_{\mathcal{H}} = \lambda_Y$. We conclude the following.

Corollary 1. *Let X_S be a solution of the E-system such that $E = 1$. Let tr be the Juyumaya trace on $Y_{d,n}(u)$ with parameters z, X_S , and let τ be the Ocneanu trace on $\mathcal{H}_n(q)$ with parameter ζ . If we take $q = u$ and $\zeta = z$, then*

$$P(\hat{\alpha}) = \Delta_S(\hat{\alpha}) \quad (\alpha \in B_n)$$

for all $n \in \mathbb{N}$.

Since the map P is invariant under the Hecke algebra automorphism (1.8), we also obtain the following.

Corollary 2. *Let X_S be a solution of the E-system such that $E = 1$. Let tr be the Juyumaya trace on $Y_{d,n}(u)$ with parameters z, X_S , and let τ be the Ocneanu trace on $\mathcal{H}_n(q)$ with parameter ζ . If we take $q = 1/u$ and $\zeta = -z/u$, then*

$$P(\hat{\alpha}) = \Delta_S(\hat{\alpha}) \quad (\alpha \in B_n)$$

for all $n \in \mathbb{N}$.

In the next sections we will explore the remaining cases where the maps P and Δ_S coincide, and show that they are all trivial, that is, either $u = 1$ or $q = 1$ or $E = 1$.

4. COMPARING P AND Δ_S

From now on, let $X_S = \{x_1, \dots, x_{d-1}\}$ be a solution of the E-system parametrized by a subset S of $\mathbb{Z}/d\mathbb{Z}$. Let tr be the Juyumaya trace on $Y_{d,n}(u)$ with parameters z, X_S , and let $E = \text{tr}(e_i) = 1/|S|$. Let τ be the Ocneanu trace on $\mathcal{H}_n(q)$ with parameter ζ . In this section, we will assume that the maps P and Δ_S coincide, and we will see what restrictions this assumption imposes on the values of q, ζ, u, z and E .

4.1. *Some equalities.* First of all, if P and Δ_S coincide, they should take the same value on the closure of any braid in any B_n . In particular for the identity braid 1 in each B_n we have:

$$P(\hat{1}) = D_{\mathcal{H}}^{n-1} = D_Y^{n-1} = \Delta_S(\hat{1})$$

for all $n \in \mathbb{N}$, whence we deduce that

$$(4.1) \quad D_{\mathcal{H}} = D_Y.$$

From (4.1) we obtain the following equality:

$$(4.2) \quad (u\zeta + z^2 - uEz + Ez)q = u\zeta(\zeta + 1).$$

If $\zeta = -1$, then we must have $u\zeta + z^2 - uEz + Ez = 0$. If $\zeta \neq -1$, then (4.2) yields the following equality for q :

$$(4.3) \quad q = \frac{u\zeta^2 + u\zeta}{u\zeta + z^2 - uEz + Ez}.$$

Now if $P = \Delta_S$ and $D_{\mathcal{H}} = D_Y$, we must have

$$(4.4) \quad \frac{\tau(\pi(\alpha))}{\text{tr}(\delta(\alpha))} = \left(\sqrt{\frac{\lambda_Y}{\lambda_{\mathcal{H}}}} \right)^{\epsilon(\alpha)} = \left(\frac{\zeta}{z} \right)^{\epsilon(\alpha)}$$

for all $\alpha \in B_n$ and for all $n \in \mathbb{N}$. Taking $\alpha = \sigma_1^2 \in B_n$ and $\alpha = \sigma_1^3 \in B_n$, for $n \geq 2$, we obtain respectively:

$$(4.5) \quad (z^2\zeta + z^2)q = \zeta(b\zeta + z^2),$$

where

$$b := \text{tr}(g_1^2) = 1 + (u-1)E + (u-1)z,$$

and

$$(4.6) \quad (bz\zeta + z^3)q = \zeta(c\zeta + bz),$$

where

$$c := \text{tr}(g_1^3) = (u^2 - u)E + (u^2 - u + 1)z.$$

Note that Equation 4.5 implies that $\zeta = -1$ if and only if $b\zeta + z^2 = 0$.

From now on, let us assume that $\zeta \neq -1$. Then (4.5) and (4.6) yield respectively the following equalities for q :

$$(4.7) \quad q = \frac{b\zeta^2 + z^2\zeta}{z^2\zeta + z^2}$$

and

$$(4.8) \quad q = \frac{c\zeta^2 + bz\zeta}{bz\zeta + z^3}.$$

Suppose first that $z^2 \neq b$. Combining (4.3) and (4.7) yields:

$$(4.9) \quad \zeta^2 = \frac{bz^2 - ubEz + bEz - uz^2}{uz^2 - ub}\zeta + \frac{z^4 - uEz^3 + Ez^3 - uz^2}{uz^2 - ub}.$$

Suppose now that $bz \neq c$. Combining (4.3) and (4.8) yields:

$$(4.10) \quad \zeta^2 = \frac{cz^2 - ucEz + cEz - uz^3}{ubz - uc}\zeta + \frac{bz^3 - ubEz^2 + bEz^2 - uz^3}{ubz - uc}.$$

Combining (4.9) and (4.10) yields $\zeta = -1$, which contradicts our assumption, unless

$$u = 1 \text{ or } E = 1 \text{ or } z = \frac{1 - E + u + uE}{2}.$$

We conclude that the only cases where the invariants P and Δ_S may coincide are the following:

- (1) $\zeta = -1$;
- (2) $\zeta \neq -1, z^2 = b$;
- (3) $\zeta \neq -1, bz = c$;
- (4) $\zeta \neq -1, u = 1$;
- (5) $\zeta \neq -1, E = 1$;
- (6) $\zeta \neq -1, z = (1 - E + u + uE)/2$.

4.2. *The case $\zeta = -1$.* If $\zeta = -1$, then Equations 4.2 and 4.5 imply that

$$z^2 = uEz - Ez + u \text{ and } z^2 = b = 1 + (u - 1)E + (u - 1)z.$$

Combining the two equalities above yields

$$(u - 1)(E + z) = (u - 1)(Ez + 1)$$

which is true only if $u = 1$ or $E = 1$ or $z = 1$.

If $u = 1$, then $z = -1$ or $z = 1$. If $E = 1$, then $z = -1$ or $z = u$. If $z = 1$ and $u \neq 1$, then $E = -1$ which is absurd.

4.3. *The case $\zeta \neq -1, z^2 = b$.* If $z^2 = b = 1 + (u - 1)E + (u - 1)z$, then Equation 4.7 yields $q = \zeta$. Replacing $q = \zeta$ in (4.3), we obtain that $z^2 = uEz - Ez + u$. As in the previous subsection, we conclude that $u = 1$ or $E = 1$ or $z = 1$.

If $u = 1$, then $z = -1$ or $z = 1$. If $E = 1$, then $z = -1$ or $z = u$. If $z = 1$ and $u \neq 1$, then $E = -1$ which is absurd.

4.4. *The case $\zeta \neq -1, bz = c$.* We have

$$bz = z + uEz - Ez + uz^2 - z^2 = u^2z - uz + z + u^2E - uE = c$$

which yields

$$z(u - 1)(E + z) = u(u - 1)(E + z).$$

In the next two subsections, we will see what happens when $u = 1$ and $E = 1$. Thus, for the moment we may assume that $u \neq 1$ and $E \neq 1$.

If $z = -E$, then $b = 1$, and combining Equations 4.3 and 4.7 yields $\zeta = \pm E$ and $q = 1$. If $z \neq -E$, then $z = u$, and combining Equations 4.3 and 4.8 yields a contradiction.

4.5. *The case $\zeta \neq -1, u = 1$.* If $\zeta \neq -1$ and $u = 1$, Equation 4.3 becomes

$$(4.11) \quad q = \frac{\zeta^2 + \zeta}{\zeta + z^2}.$$

Moreover, we have $b = 1$, so Equation 4.7 becomes

$$(4.12) \quad q = \frac{\zeta^2 + z^2\zeta}{z^2\zeta + z^2}.$$

Combining Equations 4.11 and 4.12 gives us

$$\zeta^2(z^2 - 1) = z^2(z^2 - 1),$$

which holds only if $\zeta = z$ or $\zeta = -z$ or $z = 1$ or $z = -1$.

If $\zeta = \pm z$, then $q = 1$. If $z = \pm 1$, then $q = \zeta$.

4.6. *The case $\zeta \neq -1$, $E = 1$.* Suppose that $z^2 \neq b$ (the case $z^2 = b$ has been completely covered in §4.3). Then Equation 4.9 becomes:

$$(4.13) \quad \zeta^2 = \frac{zu - z}{u}\zeta + \frac{z^2}{u}.$$

The above quadratic equation has two solutions: $\zeta = z$ or $\zeta = -z/u$. If $\zeta = z$, then Equation 4.3 yields $q = u$. If $\zeta = -z/u$, then Equation 4.3 yields $q = 1/u$.

4.7. *The case $\zeta \neq -1$, $z = (1 - E + u + uE)/2$.* As in the previous subsection, we may assume that $z^2 \neq b$. Then Equation 4.9 becomes:

$$(4.14) \quad \zeta^2 = \frac{-1 + E - 2u - Eu^2 - u^2}{2u}\zeta + \frac{-1 + 2E - 2u - 2Eu^2 - E^2 - u^2 + 2E^2u - E^2u^2}{4u}.$$

The above quadratic equation has two solutions: $\zeta = -z$ or $\zeta = -z/u$. If $\zeta = -z$, then Equation 4.3 yields $q = -u$. If $\zeta = -z/u$, then Equation 4.3 yields $q = -1/u$.

4.8. *Summarizing.* The cases below are the only cases where $D_{\mathcal{H}} = D_Y$ and

$$\frac{\tau(G_i^m)}{\text{tr}(g_i^m)} = \left(\frac{\zeta}{z}\right)^m \quad \text{for } m \in \{2, 3\}.$$

One can easily check, using (1.4), (1.5), (2.9) and (2.10), that the above equality holds for all $m \in \mathbb{N}$.

Case	q	ζ	u	z	E
1	1	z	1	\mathbb{C}^*	any
2	1	$-z$	1	\mathbb{C}^*	any
3	\mathbb{C}^*	q	1	1	any
4	\mathbb{C}^*	q	1	-1	any
5	\mathbb{C}^*	-1	1	1	any
6	\mathbb{C}^*	-1	1	-1	any
7	1	E	\mathbb{C}^*	$-E$	any
8	1	$-E$	\mathbb{C}^*	$-E$	any
9	\mathbb{C}^*	q	\mathbb{C}^*	-1	1
10	\mathbb{C}^*	q	\mathbb{C}^*	u	1
11	\mathbb{C}^*	-1	\mathbb{C}^*	-1	1
12	\mathbb{C}^*	-1	\mathbb{C}^*	u	1
13	u	z	\mathbb{C}^*	\mathbb{C}^*	1
14	$1/u$	$-z/u$	\mathbb{C}^*	\mathbb{C}^*	1
15	$-u$	$-z$	\mathbb{C}^*	$(1 - E + u + uE)/2$	any
16	$-1/u$	$-z/u$	\mathbb{C}^*	$(1 - E + u + uE)/2$	any

4.9. *Dismissing two cases.* Let us now take $\alpha = \sigma_1\sigma_2^2\sigma_1\sigma_2^2$ in B_n for $n \geq 3$. We have

$$\tau(\pi(\alpha)) = \tau(G_1G_2^2G_1G_2^2) = (q^2\zeta - 2q\zeta + \zeta)(q^2\zeta - q\zeta + \zeta + q^2 - q) + (2q^2\zeta - 2q\zeta + q^2)(q\zeta - \zeta + q).$$

and

$$\begin{aligned} \text{tr}(\delta(\alpha)) &= \text{tr}(g_1g_2^2g_1g_2^2) = \text{tr}(g_1^2) + 2(u-1)\text{tr}(g_1^2e_2) + 2(u-1)\text{tr}(g_1^2g_2e_2) \\ &\quad (u-1)^2[\text{tr}(g_1e_2g_1e_2) + 2\text{tr}(g_1e_2g_1g_2e_2) + \text{tr}(g_1g_2e_2g_1g_2e_2)]. \end{aligned}$$

Here, the fact that $X_S = \{x_1, \dots, x_{d-1}\}$ is a solution of the E-system simplifies calculations a lot. For example, we automatically deduce that $\text{tr}(g_1^2e_2) \stackrel{\text{E}}{=} E \text{tr}(g_1^2)$. Now let us see what happens when

we try to calculate $\text{tr}(g_1 e_2 g_1 e_2)$. This is always equal to:

$$\frac{1}{d^2} \sum_{k=0}^{d-1} \sum_{m=0}^{d-1} \text{tr}(g_1 t_2^k t_3^{-k} g_1 t_2^m t_3^{-m}) = \frac{1}{d^2} \sum_{k=0}^{d-1} \sum_{m=0}^{d-1} \text{tr}(g_1^2 t_1^k t_2^m t_3^{-k-m}) = \frac{1}{d^2} \sum_{k=0}^{d-1} \sum_{m=0}^{d-1} x_{d-k-m} \text{tr}(g_1^2 t_1^k t_2^m)$$

We have

$$\begin{aligned} \text{tr}(g_1^2 t_1^k t_2^m) &= \text{tr}(t_1^k t_2^m) + (u-1) \text{tr}(e_1 t_1^k t_2^m) + (u-1) \text{tr}(g_1 e_1 t_1^k t_2^m) \\ &= x_k x_m + (u-1) \frac{1}{d} \sum_{l=0}^{d-1} x_{k+l} x_{m-l} + (u-1) z x_{k+m} \\ &\stackrel{\text{E}}{=} x_k x_m + (u-1) E x_{k+m} + (u-1) z x_{k+m} \end{aligned}$$

Thus,

$$\begin{aligned} \text{tr}(g_1 e_2 g_1 e_2) &= \frac{1}{d^2} \sum_{k=0}^{d-1} \sum_{m=0}^{d-1} x_{d-k-m} x_k x_m + [(u-1)(E+z)] \frac{1}{d^2} \sum_{k=0}^{d-1} \sum_{m=0}^{d-1} x_{d-k-m} x_{k+m} \\ &\stackrel{\text{E}}{=} \frac{1}{d} \sum_{k=0}^{d-1} x_{d-k} x_k E + [(u-1)(E+z)] \frac{1}{d} \sum_{k=0}^{d-1} E \\ &\stackrel{\text{E}}{=} E^2 + [(u-1)(E+z)] E = E(uE + uz - z) = E \text{tr}(e_1 g_1^2) \end{aligned}$$

We finally obtain that

$$\text{tr}(g_1 g_2^2 g_1 g_2^2) \stackrel{\text{E}}{=} b(2b-1) + (u-1)^2 (E + uz + z)(uE + uz - z) + u(u-1)^2 z^2.$$

In Cases 1–14, we obtain:

$$\frac{\tau(\pi(\alpha))}{\text{tr}(\delta(\alpha))} = \frac{\tau(G_1 G_2^2 G_1 G_2^2)}{\text{tr}(g_1 g_2^2 g_1 g_2^2)} \stackrel{\text{E}}{=} \left(\frac{\zeta}{z}\right)^6 = \left(\frac{\zeta}{z}\right)^{\epsilon(\alpha)}.$$

Cases 15 and 16 collapse, unless $E = 1$ or $u = \pm 1$. However:

- Case 15, $E = 1$ is covered by Case 10;
- Case 15, $u = 1$ is covered by Case 3;
- Case 15, $u = -1$ is covered by Case 7;
- Case 16, $E = 1$ is covered by Case 12;
- Case 16, $u = 1$ is covered by Case 3;
- Case 16, $u = -1$ is covered by Case 8.

5. THE ONLY CASES WHERE P AND Δ_S COINCIDE

We will show that the cases where the invariants P and Δ_S coincide are precisely the Cases 1–14 in the table of §4.8. We have already shown that if P and Δ_S coincide, then we must be in one of these cases. We will now show that in all these cases, P and Δ_S do coincide. Note that these results hold for any $d \in \mathbb{N}$ and for any non-empty subset S of $\mathbb{Z}/d\mathbb{Z}$.

5.1. General methodology. We already know (see §4.8) that, for $\alpha = \sigma_i^m \in B_n$ ($1 \leq i \leq n-1$), in Cases 1–14 we have

$$(5.1) \quad D_{\mathcal{H}} = D_Y \quad \text{and} \quad \frac{\tau(\pi(\sigma_i^m))}{\text{tr}(\delta(\sigma_i^m))} = \left(\frac{\zeta}{z}\right)^m \quad \text{for all } m \in \mathbb{N}.$$

In particular, following (1.4), (1.5), (2.9) and (2.10), in Cases 3–6 and 9–12 we have

$$(5.2) \quad \tau(\pi(\sigma_i^m)) = \zeta^m \quad \text{and} \quad \text{tr}(\delta(\sigma_i^m)) = z^m \quad \text{for all } m \in \mathbb{N}.$$

We need to show that

$$(5.3) \quad \frac{\tau(\pi(\alpha))}{\text{tr}(\delta(\alpha))} = \left(\frac{\zeta}{z}\right)^{\epsilon(\alpha)} \quad \text{for all } \alpha \in B_n,$$

where $\epsilon(\alpha)$ is the sum of the exponents of the braid generators in the braid word α . In Cases 3–6 and 9–12 we will even show that

$$(5.4) \quad \tau(\pi(\alpha)) = \zeta^{\epsilon(\alpha)} \quad \text{for all } \alpha \in B_n$$

and

$$(5.5) \quad \text{tr}(\delta(\alpha)) = z^{\epsilon(\alpha)} \quad \text{for all } \alpha \in B_n.$$

We will prove the above by induction on the following non-negative number:

$$\nu(\alpha) := |\text{sum of all negative exponents of the braid generators in } \alpha|.$$

Note that $\nu(\alpha)$ depends on the expression of α in terms of braid generators.

For the inductive step, we will show (5.3) in Cases 7–8 and 13–14 (respectively (5.4) and (5.5) in Cases 3–6 and 9–12) for $\alpha\sigma_i^{-1}$ ($\alpha \in B_n$, $1 \leq i \leq n-1$) using the induction hypothesis. The formulas for $\pi(\sigma_i^{-1}) = G_i^{-1}$ and $\delta(\sigma_i^{-1}) = g_i^{-1}$ are given respectively by Equations 1.3 and 2.4.

For the first step ($\nu(\alpha) = 0$), we will proceed by induction on n . Set

$$B_n^+ := \{\alpha \in B_n \mid \nu(\alpha) = 0\}^2$$

The cases $n = 1$ and $n = 2$ are taken care of by (5.1) for Cases 7–8 and 13–14 (respectively (5.2) for Cases 3–6 and 9–12). We will only need to prove (5.3) (respectively (5.4) and (5.5)) for B_{n+1}^+ assuming that it holds for B_n^+ . To do this we will use a second induction on the non-negative number $\epsilon(\beta) + \epsilon_n(\beta)$, where $\epsilon_n(\beta)$ denotes the sum of the exponents of the braid generator σ_n in the braid word $\beta \in B_{n+1}^+$. Note that $\epsilon(\beta)$ is uniquely defined for β , whereas $\epsilon_n(\beta)$ depends on the expression of β in terms of braid generators. However, we always have $\epsilon(\beta) \geq \epsilon_n(\beta)$.

If $\epsilon(\beta) + \epsilon_n(\beta) = 0$, then $\beta = 1$ and there is nothing to prove. If $\epsilon(\beta) + \epsilon_n(\beta) = 1$, then $\epsilon(\beta) = 1$ and $\epsilon_n(\beta) = 0$. Hence, $\beta = \sigma_i$ for some $1 \leq i \leq n-1$ and the desired result holds. Now assume that $\epsilon(\beta) + \epsilon_n(\beta) > 1$ and that the result holds for smaller values of $\epsilon + \epsilon_n$. We will distinguish three cases:

- If $\epsilon_n(\beta) = 0$, then $\beta \in B_n^+$, and the induction hypothesis on n yields the desired result.
- If $\epsilon_n(\beta) = 1$, then there exist $\alpha_1, \alpha_2 \in B_n^+$ such that $\beta = \alpha_1\sigma_n\alpha_2$. We have

$$\tau(\pi(\beta)) = \zeta \tau(\pi(\alpha_1\alpha_2)) \quad \text{and} \quad \text{tr}(\delta(\beta)) = z \text{tr}(\delta(\alpha_1\alpha_2)).$$

The induction hypothesis on n yields the rest.

- If $\epsilon_n(\beta) \geq 2$, then there exist $\alpha \in B_n^+$ and $\beta_1, \beta_2 \in B_{n+1}^+$ such that $\beta = \beta_1\sigma_n\alpha\sigma_n\beta_2$. We will need the following lemma:

Lemma 3. *Let $\alpha \in B_n^+$. Then one of the following hold:*

- (i) $\sigma_n\alpha\sigma_n = \alpha_1\sigma_n\alpha_2$, for some $\alpha_1, \alpha_2 \in B_n^+$, or
- (ii) $\sigma_n\alpha\sigma_n = \beta_1\sigma_j^2\beta_2$, for some $\beta_1, \beta_2 \in B_{n+1}^+$ and $1 \leq j \leq n$.

Proof. We will proceed by induction on n . If $n = 1$, then $\alpha = 1$ and we are in Case (ii). Assume that the above holds for $1, 2, \dots, n-1$. We will show that it holds for n . We will proceed by induction on the number $\epsilon_{n-1}(\alpha)$, that is, the sum of the exponents of the braid generator σ_{n-1} in the braid word α :

- If $\epsilon_{n-1}(\alpha) = 0$, then σ_n commutes with α , and

$$\sigma_n\alpha\sigma_n = \alpha\sigma_n^2.$$

² This set is known as the *braid monoid*, see, for example, [GePf, Chapter 4].

- If $\epsilon_{n-1}(\alpha) = 1$, then there exist $b_1, b_2 \in B_{n-1}^+$ such that $\alpha = b_1 \sigma_{n-1} b_2$. We have:

$$\sigma_n \alpha \sigma_n = b_1 \sigma_n \sigma_{n-1} \sigma_n b_2 = b_1 \sigma_{n-1} \sigma_n \sigma_{n-1} b_2 = \alpha_1 \sigma_n \alpha_2,$$

where $\alpha_1 = b_1 \sigma_{n-1} \in B_n^+$ and $\alpha_2 = \sigma_{n-1} b_2 \in B_n^+$.

- If $\epsilon_{n-1}(\alpha) \geq 2$, then there exist $b \in B_{n-1}^+$ and $\alpha_1, \alpha_2 \in B_n^+$ such that $\alpha = \alpha_1 \sigma_{n-1} b \sigma_{n-1} \alpha_2$. Then, by the induction hypothesis on n , one of the following hold:

- (i) $\sigma_{n-1} b \sigma_{n-1} = b_1 \sigma_{n-1} b_2$, for some $b_1, b_2 \in B_{n-1}^+$, or
- (ii) $\sigma_{n-1} b \sigma_{n-1} = \alpha'_1 \sigma_j^2 \alpha'_2$, for some $\alpha'_1, \alpha'_2 \in B_n^+$ and $1 \leq j \leq n-1$.

In Case (i), the induction hypothesis on $\epsilon_{n-1}(\alpha)$ yields the desired result. In Case (ii), we obtain:

$$\sigma_n \alpha \sigma_n = \sigma_n \alpha_1 \alpha'_1 \sigma_j^2 \alpha'_2 \alpha_2 \sigma_n = \beta_1 \sigma_j^2 \beta_2,$$

where $\beta_1 = \sigma_n \alpha_1 \alpha'_1 \in B_{n+1}^+$ and $\beta_2 = \alpha'_2 \alpha_2 \sigma_n \in B_{n+1}^+$.

□

Applying now the above lemma to the word $\beta = \beta_1 \sigma_n \alpha \sigma_n \beta_2$, where $\alpha \in B_n^+$ and $\beta_1, \beta_2 \in B_{n+1}^+$, we obtain that one of the following hold:

- (i) $\beta = \beta_1 \alpha_1 \sigma_n \alpha_2 \beta_2$, for some $\alpha_1, \alpha_2 \in B_n^+$, or
- (ii) $\beta = \beta_1 \beta'_1 \sigma_j^2 \beta'_2 \beta_2$, for some $\beta'_1, \beta'_2 \in B_{n+1}^+$ and $1 \leq j \leq n$.

The induction hypothesis on $\epsilon(\beta) + \epsilon_n(\beta)$ covers Case (i), since $\epsilon_n(\beta_1 \alpha_1 \sigma_n \alpha_2 \beta_2) < \epsilon_n(\beta_1 \sigma_n \alpha \sigma_n \beta_2)$. Therefore, it is enough to prove (5.3) (respectively (5.4) and (5.5)) in Case (ii). Since τ and tr are trace functions, we deduce that we will check the desired equalities on all words of the form:

$$\beta \sigma_j^2 \quad (\beta \in B_{n+1}^+, 1 \leq j \leq n).$$

To summarize: In order to prove Equality 5.3 in Cases 7–8 and 13–14, and Equalities 5.4 and 5.5 in Cases 3–6 and 9–12, we will show that these equalities hold on all words of the form

$$\beta \sigma_j^2 \quad (\beta \in B_{n+1}^+, 1 \leq j \leq n),$$

assuming the induction hypotheses on n and on $\epsilon + \epsilon_n$, and all words of the form

$$\alpha \sigma_i^{-1} \quad (\alpha \in B_n, 1 \leq i \leq n-1),$$

assuming the induction hypothesis on ν .

5.2. The Cases 1 and 2. In the first two cases, although our general methodology applies, we prefer to use the following, simpler approach. Since $q = 1$, the quadratic relation (1.2)(h) in the Iwahori-Hecke algebra $\mathcal{H}_n(1) \cong \mathfrak{S}_n$ becomes $G_i^2 = 1$. Similarly, since $u = 1$, the quadratic relation (2.2) in the Yokonuma-Hecke algebra $Y_{d,n}(1)$ becomes $g_i^2 = 1$. Therefore, there exist two natural isomorphisms ι^+ and ι^- between $\pi(\mathbb{C}B_n) \cong \mathcal{H}_n(1)$ and $\delta(\mathbb{C}B_n)$ given by $\iota^+(G_i) = g_i$ and $\iota^-(G_i) = -g_i$ respectively. Now, if we take $\zeta = z$, then $\text{tr} \circ \iota^+$ is a Markov trace on $\mathcal{H}_n(1)$ that satisfies all three rules of Theorem 2. The uniqueness of the Ocneanu trace yields $\text{tr} \circ \iota^+ = \tau$. So in Case 1 we have:

$$\frac{\tau(\pi(\alpha))}{\text{tr}(\delta(\alpha))} = \frac{\text{tr}(\iota^+(\pi(\alpha)))}{\text{tr}(\delta(\alpha))} = \frac{\text{tr}(\delta(\alpha))}{\text{tr}(\delta(\alpha))} = 1 = \left(\frac{z}{z}\right)^{\epsilon(\alpha)} = \left(\frac{\zeta}{z}\right)^{\epsilon(\alpha)} \quad \text{for all } \alpha \in B_n.$$

Similarly, if we take $\zeta = -z$, then $\text{tr} \circ \iota^-$ is a Markov trace that satisfies all three rules of Theorem 2. Therefore, we obtain $\text{tr} \circ \iota^- = \tau$. So in Case 2 we have:

$$\frac{\tau(\pi(\alpha))}{\text{tr}(\delta(\alpha))} = \frac{\text{tr}(\iota^-(\pi(\alpha)))}{\text{tr}(\delta(\alpha))} = \frac{(-1)^{\epsilon(\alpha)} \text{tr}(\delta(\alpha))}{\text{tr}(\delta(\alpha))} = (-1)^{\epsilon(\alpha)} = \left(\frac{-z}{z}\right)^{\epsilon(\alpha)} = \left(\frac{\zeta}{z}\right)^{\epsilon(\alpha)} \quad \text{for all } \alpha \in B_n.$$

5.3. *The Cases 3–6.* Following our general methodology, let first $\beta \in B_{n+1}^+$ and $1 \leq j \leq n$. If $\zeta = q$, we have

$$\tau(\pi(\beta\sigma_j^2)) = (q-1)\tau(\pi(\beta\sigma_j)) + q\tau(\pi(\beta)) \stackrel{\text{ind. hyp.}}{=} (q-1)q^{\epsilon(\beta)+1} + q \cdot q^{\epsilon(\beta)} = q^{\epsilon(\beta)+2} = \zeta^{\epsilon(\beta\sigma_j^2)}.$$

If $\zeta = -1$, we have

$$\tau(\pi(\beta\sigma_j^2)) = (q-1)\tau(\pi(\beta\sigma_j)) + q\tau(\pi(\beta)) \stackrel{\text{ind. hyp.}}{=} (q-1)(-1)^{\epsilon(\beta)+1} + q(-1)^{\epsilon(\beta)} = (-1)^{\epsilon(\beta)+2} = \zeta^{\epsilon(\beta\sigma_j^2)}.$$

If $u = 1$ and $z = 1$, we have

$$\text{tr}(\delta(\beta\sigma_j^2)) = \text{tr}(\delta(\beta)) \stackrel{\text{ind. hyp.}}{=} 1 = z^{\epsilon(\beta\sigma_j^2)}.$$

If $u = 1$ and $z = -1$, we have

$$\text{tr}(\delta(\beta\sigma_j^2)) = \text{tr}(\delta(\beta)) \stackrel{\text{ind. hyp.}}{=} (-1)^{\epsilon(\beta)} = (-1)^{\epsilon(\beta)+2} = z^{\epsilon(\beta\sigma_j^2)}.$$

Now let $\alpha \in B_n$ and $1 \leq i \leq n-1$. If $\zeta = q$, we have

$$\tau(\pi(\alpha\sigma_i^{-1})) = q^{-1}\tau(\pi(\alpha\sigma_i)) + (q^{-1}-1)\tau(\pi(\alpha)) \stackrel{\text{ind. hyp.}}{=} q^{-1}q^{\epsilon(\alpha)+1} + (q^{-1}-1)q^{\epsilon(\alpha)} = q^{\epsilon(\alpha)-1} = \zeta^{\epsilon(\alpha\sigma_i^{-1})}.$$

If $\zeta = -1$, we have

$$\tau(\pi(\alpha\sigma_i^{-1})) = q^{-1}\tau(\pi(\alpha\sigma_i)) + (q^{-1}-1)\tau(\pi(\alpha)) \stackrel{\text{ind. hyp.}}{=} q^{-1}(-1)^{\epsilon(\alpha)+1} + (q^{-1}-1)(-1)^{\epsilon(\alpha)} = (-1)^{\epsilon(\alpha)-1} = \zeta^{\epsilon(\alpha\sigma_i^{-1})}.$$

If $u = 1$ and $z = 1$, we have

$$\text{tr}(\delta(\alpha\sigma_i^{-1})) = \text{tr}(\delta(\alpha\sigma_i)) \stackrel{\text{ind. hyp.}}{=} 1 = z^{\epsilon(\alpha\sigma_i^{-1})}.$$

If $u = 1$ and $z = -1$, we have

$$\text{tr}(\delta(\alpha\sigma_i^{-1})) = \text{tr}(\delta(\alpha\sigma_i)) \stackrel{\text{ind. hyp.}}{=} (-1)^{\epsilon(\alpha)+1} = (-1)^{\epsilon(\alpha)-1} = z^{\epsilon(\alpha\sigma_i^{-1})}.$$

Thus, we conclude that (5.4) and (5.5) hold, whence we deduce (5.3).

5.4. *The Cases 7 and 8.* In order to show (5.3) in Cases 7 and 8, we will first show that

$$(5.6) \quad \tau(hG_j^2) = \tau(h) \quad (h \in \mathcal{H}_n(1), 1 \leq j \leq n-1)$$

and

$$(5.7) \quad \text{tr}(yg_j^2) = \text{tr}(y) \quad (y \in Y_{d,n}(u), 1 \leq j \leq n-1).$$

Note that (5.7) is equivalent to

$$(5.8) \quad \text{tr}(ye_j) = -\text{tr}(yg_j e_j) \quad (y \in Y_{d,n}(u), 1 \leq j \leq n-1).$$

Equation 5.6 is straightforward, since $G_j^2 = 1$, for all $j = 1, \dots, n-1$, in $\mathcal{H}_n(1)$. To prove (5.8) we will proceed by induction on n . Recall that $z = -E$. It is enough to show that (5.8) holds on the elements of the inductive basis of $Y_{d,n}(u)$.

Let $n = 2$. Let $y = t_1^k g_1 t_1^l$ for some $k, l \in \mathbb{Z}/d\mathbb{Z}$. Then, by (2.6) and (2.12), we have

$$\text{tr}(ye_1) = \text{tr}(t_1^k g_1 t_1^l e_1) = \text{tr}(t_1^{k+l} g_1 e_1) = -E \text{tr}(t_1^{k+l}) = -E x_{k+l},$$

and, by (2.1)(f₂), Lemma 1 and (2.15), we have

$$\text{tr}(yg_1 e_1) = \text{tr}(t_1^k g_1 t_1^l g_1 e_1) = \text{tr}(t_1^k g_1^2 e_1 t_1^l) = \text{tr}(t_1^{k+l} e_1) + (u-1) \text{tr}(t_1^{k+l} e_1) + (u-1) \text{tr}(t_1^{k+l} g_1 e_1) \stackrel{E}{=} E x_{k+l}.$$

So (5.8) holds. Now let $y = t_1^k t_2^l$ for some $k, l \in \mathbb{Z}/d\mathbb{Z}$. Then, by (2.1)(f₂), (2.6) and Lemma 1, we have

$$\text{tr}(ye_1) = \text{tr}(t_1^k t_2^l e_1) = \text{tr}(t_1^{k+l} e_1) = E x_{k+l}$$

and

$$\text{tr}(yg_1 e_1) = \text{tr}(t_1^k t_2^l g_1 e_1) = \text{tr}(t_1^k g_1 e_1 t_1^l) = -E x_{k+l}.$$

So (5.8) holds.

Now assume that (5.8) holds for n . We will prove it for $n+1$. Let $y = w_n g_n g_{n-1} \dots g_i t_i^k$ for some $k \in \mathbb{Z}/d\mathbb{Z}$ and some $w_n \in Y_{d,n}(u)$. If $j < n$, then, following the definition of the trace and the induction hypothesis, we have

$$\mathrm{tr}(y e_j) = \mathrm{tr}(w_n g_n g_{n-1} \dots g_i t_i^k e_j) = -E \mathrm{tr}(w_n g_{n-1} \dots g_i t_i^k e_j) = E \mathrm{tr}(w_n g_{n-1} \dots g_i t_i^k g_j e_j) = -\mathrm{tr}(y g_j e_j).$$

If $j = n$, then, we have to distinguish two cases: If $i = n$, then we have, by (2.6),

$$\mathrm{tr}(y e_n) = \mathrm{tr}(w_n g_n t_n^k e_n) = \mathrm{tr}(t_n^k w_n g_n e_n) = -E \mathrm{tr}(t_n^k w_n),$$

and, by (2.1)(f₂), (2.6) and Lemma 1,

$$\mathrm{tr}(y g_n e_n) = \mathrm{tr}(w_n g_n t_n^k g_n e_n) = \mathrm{tr}(t_n^k w_n g_n^2 e_n) = u \mathrm{tr}(t_n^k w_n e_n) + (u-1) \mathrm{tr}(t_n^k w_n g_n e_n) = E \mathrm{tr}(t_n^k w_n).$$

If $i < n$, then

$$\mathrm{tr}(y e_n) = \mathrm{tr}(w_n g_n g_{n-1} \dots g_i t_i^k e_n) \stackrel{(2.6)}{=} \mathrm{tr}(w_n e_{n-1} g_n g_{n-1} \dots g_i t_i^k) = -E \mathrm{tr}(g_{n-1} \dots g_i t_i^k w_n e_{n-1}).$$

Following the induction hypothesis, the latter is equal to

$$\begin{aligned} E \mathrm{tr}(g_{n-1} \dots g_i t_i^k w_n g_{n-1} e_{n-1}) &= -\mathrm{tr}(w_n g_{n-1} e_{n-1} g_n g_{n-1} \dots g_i t_i^k) \stackrel{(2.6)}{=} \\ &= -\mathrm{tr}(w_n g_{n-1} g_n g_{n-1} \dots g_i t_i^k e_n) = -\mathrm{tr}(w_n g_n g_{n-1} g_n g_{n-2} \dots g_i t_i^k e_n) = -\mathrm{tr}(y g_n e_n). \end{aligned}$$

Finally, let $y = w_n t_{n+1}^k$ for some $k \in \mathbb{Z}/d\mathbb{Z}$ and some $w_n \in Y_{d,n}(u)$. If $j < n$, then, following the definition of the trace and the induction hypothesis, we have

$$\mathrm{tr}(y e_j) = \mathrm{tr}(w_n t_{n+1}^k e_j) = x_k \mathrm{tr}(w_n e_j) = -x_k \mathrm{tr}(w_n g_j e_j) = -\mathrm{tr}(y g_j e_j).$$

If $j = n$, then, with repeated use of Lemma 1 we obtain:

$$\mathrm{tr}(y e_n) = \mathrm{tr}(w_n t_{n+1}^k e_n) = \mathrm{tr}(w_n t_n^k e_n) \stackrel{E}{=} E \mathrm{tr}(w_n t_n^k) = -\mathrm{tr}(w_n t_n^k e_n g_n) = -\mathrm{tr}(w_n t_{n+1}^k e_n g_n) = -\mathrm{tr}(y g_n e_n).$$

Equations 5.6 and 5.7 imply the following for the inverses of the generators:

$$(5.9) \quad \tau(h G_j^{-1}) = \tau(h G_j) \quad (h \in \mathcal{H}_n(1), 1 \leq j \leq n-1)$$

and

$$(5.10) \quad \mathrm{tr}(y g_j^{-1}) = \mathrm{tr}(y g_j) \quad (y \in Y_{d,n}(u), 1 \leq j \leq n-1).$$

We are now ready to prove (5.3). Let $\beta \in B_{n+1}^+$ and $1 \leq j \leq n$. Following Equations 5.6 and 5.7, we obtain

$$\frac{\tau(\pi(\beta \sigma_j^2))}{\mathrm{tr}(\delta(\beta \sigma_j^2))} = \frac{\tau(\pi(\beta) G_j^2)}{\mathrm{tr}(\delta(\beta) g_j^2)} = \frac{\tau(\pi(\beta))}{\mathrm{tr}(\delta(\beta))} \stackrel{\text{ind. hyp.}}{=} \left(\frac{\zeta}{z}\right)^{\epsilon(\beta)} = \left(\frac{\zeta}{z}\right)^{\epsilon(\beta)+2} = \left(\frac{\zeta}{z}\right)^{\epsilon(\beta \sigma_j^2)},$$

since

$$(5.11) \quad \frac{\zeta}{z} = -1 \text{ in Case 7 and } \frac{\zeta}{z} = 1 \text{ in Case 8.}$$

Now let $\alpha \in B_n$ and $1 \leq i \leq n-1$. Following Equations 5.9 and 5.10, we obtain

$$\frac{\tau(\pi(\alpha \sigma_i^{-1}))}{\mathrm{tr}(\delta(\alpha \sigma_i^{-1}))} = \frac{\tau(\pi(\alpha) G_i^{-1})}{\mathrm{tr}(\delta(\alpha) g_i^{-1})} = \frac{\tau(\pi(\alpha) G_i)}{\mathrm{tr}(\delta(\alpha) g_i)} \stackrel{\text{ind. hyp.}}{=} \left(\frac{\zeta}{z}\right)^{\epsilon(\alpha)+1} = \left(\frac{\zeta}{z}\right)^{\epsilon(\alpha)-1} = \left(\frac{\zeta}{z}\right)^{\epsilon(\alpha \sigma_i^{-1})},$$

again because of (5.11). Therefore, in both Cases 7 and 8, our general methodology yields (5.3).

5.5. *The Cases 9–12.* We will show that, in these cases, (5.4) and (5.5) hold again for all $\alpha \in B_n$. On the Hecke algebra side, we have already shown in §5.3 that (5.4) holds when $\zeta = q$ or $\zeta = -1$. Thus, it remains to show (5.5) for Cases 9–12.

Let $\beta \in B_{n+1}^+$ and $1 \leq j \leq n$. We have

$$\text{tr}(\delta(\beta\sigma_j^2)) = \text{tr}(\delta(\beta)g_j^2) = \text{tr}(\delta(\beta)) + (u-1)\text{tr}(\delta(\beta)e_j) + (u-1)\text{tr}(\delta(\beta)e_jg_j).$$

By Proposition 1, if $E = 1$, the latter is equal to

$$\text{tr}(\delta(\beta)) + (u-1)\text{tr}(\delta(\beta)) + (u-1)\text{tr}(\delta(\beta)g_j).$$

On the one hand, if $z = u$, we deduce, using the induction hypothesis, that

$$\text{tr}(\delta(\beta\sigma_j^2)) = u^{\epsilon(\beta)} + (u-1)u^{\epsilon(\beta)} + (u-1)u^{\epsilon(\beta)+1} = u^{\epsilon(\beta)+2} = z^{\epsilon(\beta\sigma_j^2)}.$$

On the other hand, if $z = -1$, we deduce, using the induction hypothesis, that

$$\text{tr}(\delta(\beta\sigma_j^2)) = (-1)^{\epsilon(\beta)} + (u-1)(-1)^{\epsilon(\beta)} + (u-1)(-1)^{\epsilon(\beta)+1} = (-1)^{\epsilon(\beta)+2} = z^{\epsilon(\beta\sigma_j^2)}.$$

Now let $\alpha \in B_n$ and $1 \leq i \leq n-1$. Using formula (2.4), we obtain

$$\text{tr}(\delta(\alpha\sigma_i^{-1})) = \text{tr}(\delta(\alpha)g_i^{-1}) = \text{tr}(\delta(\alpha)g_i) + (u^{-1}-1)\text{tr}(\delta(\alpha)e_i) + (u^{-1}-1)\text{tr}(\delta(\alpha)e_ig_i).$$

By Proposition 1, if $E = 1$, the latter is equal to

$$\text{tr}(\delta(\alpha)g_i) + (u^{-1}-1)\text{tr}(\delta(\alpha)) + (u^{-1}-1)\text{tr}(\delta(\alpha)g_i).$$

On the one hand, if $z = u$, we deduce, using the induction hypothesis, that

$$\text{tr}(\delta(\alpha\sigma_i^{-1})) = u^{\epsilon(\alpha)+1} + (u^{-1}-1)u^{\epsilon(\alpha)} + (u^{-1}-1)u^{\epsilon(\alpha)+1} = u^{\epsilon(\alpha)-1} = z^{\epsilon(\alpha\sigma_i^{-1})}.$$

On the other hand, if $z = -1$, we deduce, using the induction hypothesis, that

$$\text{tr}(\delta(\alpha\sigma_i^{-1})) = (-1)^{\epsilon(\alpha)+1} + (u^{-1}-1)(-1)^{\epsilon(\alpha)} + (u^{-1}-1)(-1)^{\epsilon(\alpha)+1} = (-1)^{\epsilon(\alpha)-1} = z^{\epsilon(\alpha\sigma_i^{-1})}.$$

Thus, we conclude that (5.5) holds. Since (5.4) also holds, we deduce that (5.3) holds.

5.6. *The Cases 13 and 14.* The Cases 13 and 14 have been covered by Corollaries 1 and 2 respectively. Nevertheless, we will see here how our general methodology applies also to these cases.

Let $\beta \in B_{n+1}^+$ and $1 \leq j \leq n$. We have

$$\tau(\pi(\beta\sigma_j^2)) = \tau(\pi(\beta)G_j^2) = (q-1)\tau(\pi(\beta)G_j) + q\tau(\pi(\beta)) = (q-1)\tau(\pi(\beta\sigma_j)) + q\tau(\pi(\beta))$$

and

$$\text{tr}(\delta(\beta\sigma_j^2)) = \text{tr}(\delta(\beta)g_j^2) = \text{tr}(\delta(\beta)) + (u-1)\text{tr}(\delta(\beta)e_j) + (u-1)\text{tr}(\delta(\beta)e_jg_j).$$

Since $E = 1$, by Proposition 1, the last equation becomes:

$$\text{tr}(\delta(\beta\sigma_j^2)) = \text{tr}(\delta(\beta)) + (u-1)\text{tr}(\delta(\beta)) + (u-1)\text{tr}(\delta(\beta)g_j) = (u-1)\text{tr}(\delta(\beta\sigma_j)) + u\text{tr}(\delta(\beta)).$$

If $q = u$ and $\zeta = z$, the induction hypothesis on (5.3) yields:

$$\tau(\pi(\beta\sigma_j^2)) = (u-1) \cdot 1^{\epsilon(\beta)+1} \cdot \text{tr}(\delta(\beta\sigma_j)) + u \cdot 1^{\epsilon(\beta)} \cdot \text{tr}(\delta(\beta)) = \text{tr}(\delta(\beta\sigma_j^2)),$$

as desired. If $q = 1/u$ and $\zeta = -z/u$, the induction hypothesis on (5.3) yields:

$$\tau(\pi(\beta\sigma_j^2)) = \left(\frac{1}{u} - 1\right) \left(\frac{-1}{u}\right)^{\epsilon(\beta)+1} \text{tr}(\delta(\beta\sigma_j)) + \frac{1}{u} \left(\frac{-1}{u}\right)^{\epsilon(\beta)} \text{tr}(\delta(\beta)) = \left(\frac{-1}{u}\right)^{\epsilon(\beta)+2} \text{tr}(\delta(\beta\sigma_j^2)),$$

as desired.

Now let $\alpha \in B_n$ and $1 \leq i \leq n-1$. We have

$$\tau(\pi(\alpha\sigma_i^{-1})) = q^{-1}\tau(\pi(\alpha\sigma_i)) + (q^{-1}-1)\tau(\pi(\alpha))$$

and

$$\mathrm{tr}(\delta(\alpha\sigma_i^{-1})) = \mathrm{tr}(\delta(\alpha)g_i^{-1}) = \mathrm{tr}(\delta(\alpha)g_i) + (u^{-1} - 1) \mathrm{tr}(\delta(\alpha)e_i) + (u^{-1} - 1) \mathrm{tr}(\delta(\alpha)g_i e_i).$$

Since $E = 1$, by Proposition 1, the last equation becomes:

$$\mathrm{tr}(\delta(\alpha\sigma_i^{-1})) = \mathrm{tr}(\delta(\alpha)g_i) + (u^{-1} - 1) \mathrm{tr}(\delta(\alpha)) + (u^{-1} - 1) \mathrm{tr}(\delta(\alpha)g_i) = u^{-1} \mathrm{tr}(\delta(\alpha\sigma_i)) + (u^{-1} - 1) \mathrm{tr}(\delta(\alpha)).$$

If $q = u$ and $\zeta = z$, the induction hypothesis on (5.3) yields:

$$\tau(\pi(\alpha\sigma_i^{-1})) = u^{-1} \cdot 1^{\epsilon(\alpha)+1} \cdot \mathrm{tr}(\delta(\alpha\sigma_i)) + (u^{-1} - 1) \cdot 1^{\epsilon(\alpha)} \cdot \mathrm{tr}(\delta(\alpha)) = \mathrm{tr}(\delta(\alpha\sigma_i^{-1})),$$

as desired. If $q = 1/u$ and $\zeta = -z/u$, the induction hypothesis on (5.3) yields:

$$\tau(\pi(\alpha\sigma_i^{-1})) = u \left(\frac{-1}{u} \right)^{\epsilon(\alpha)+1} \mathrm{tr}(\delta(\alpha\sigma_i)) + (u - 1) \left(\frac{-1}{u} \right)^{\epsilon(\alpha)} \mathrm{tr}(\delta(\alpha)) = \left(\frac{-1}{u} \right)^{\epsilon(\alpha)-1} \mathrm{tr}(\delta(\alpha\sigma_i^{-1})),$$

as desired. Following our general methodology, (5.3) holds also in Cases 13 and 14.

5.7. Conclusion. The following result, proved in Subsections 5.1–5.6, is the main result of this paper.

Theorem 5. *Let X_S be a solution of the E-system. Let tr be the Juyumaya trace on $Y_{d,n}(u)$ with parameters z, X_S , and let τ be the Ocneanu trace on $\mathcal{H}_n(q)$ with parameter ζ . Let $E = \mathrm{tr}(e_i)$ for all $i = 1, \dots, n-1$. Then $P = \Delta_S$ if and only if we are in one of the cases portrayed in the following table:*

Case	q	ζ	u	z	E
1	1	z	1	\mathbb{C}^*	any
2	1	$-z$	1	\mathbb{C}^*	any
3	\mathbb{C}^*	q	1	1	any
4	\mathbb{C}^*	q	1	-1	any
5	\mathbb{C}^*	-1	1	1	any
6	\mathbb{C}^*	-1	1	-1	any
7	1	E	\mathbb{C}^*	$-E$	any
8	1	$-E$	\mathbb{C}^*	$-E$	any
9	\mathbb{C}^*	q	\mathbb{C}^*	-1	1
10	\mathbb{C}^*	q	\mathbb{C}^*	u	1
11	\mathbb{C}^*	-1	\mathbb{C}^*	-1	1
12	\mathbb{C}^*	-1	\mathbb{C}^*	u	1
13	u	z	\mathbb{C}^*	\mathbb{C}^*	1
14	$1/u$	$-z/u$	\mathbb{C}^*	\mathbb{C}^*	1

5.8. Comparing further P and Δ_S . In Theorem 5 we give a necessary and sufficient condition for the invariants P and Δ_S to coincide. However, as we mentioned in the introduction, computational data do not indicate that one invariant is topologically stronger than the other. A simple explanation would be that P is a scalar multiple of Δ_S , that is, there exist $(c_n)_{n \in \mathbb{N}}$ in $\mathbb{C}(q, \zeta, u, z, E)$ such that

$$(5.12) \quad P(\hat{\alpha}) = c_n \Delta_S(\hat{\alpha}) \quad (\alpha \in B_n)$$

for all $n \in \mathbb{N}$. Then for the identity braid 1 in each B_n we have:

$$P(\hat{1}) = D_{\mathcal{H}}^{n-1} = c_n D_Y^{n-1} = c_n \Delta_S(\hat{1}).$$

We deduce that

$$c_n = \frac{D_{\mathcal{H}}^{n-1}}{D_Y^{n-1}} = \left(\frac{D_{\mathcal{H}}}{D_Y} \right)^{n-1}.$$

Thus, if (5.12) holds, we must have

$$(5.13) \quad \frac{\tau(\pi(\alpha))}{\text{tr}(\delta(\alpha))} = \left(\frac{\zeta}{z}\right)^{\epsilon(\alpha)}$$

for all $\alpha \in B_n$ and for all $n \in \mathbb{N}$. Taking $\alpha = \sigma_1^{-1} \in B_n$, for $n \geq 2$, we obtain

$$(5.14) \quad (u\zeta + z^2 - uEz + Ez)q = u\zeta(\zeta + 1),$$

which in turn yields (see §4.1)

$$D_{\mathcal{H}} = D_Y.$$

We conclude that $c_n = 1$ for all $n \in \mathbb{N}$. Combining this with Theorem 5, we obtain the following result:

Theorem 6. *Let X_S be a solution of the E-system. Let tr be the Juyumaya trace on $Y_{d,n}(u)$ with parameters z, X_S , and let τ be the Ocneanu trace on $\mathcal{H}_n(q)$ with parameter ζ . Let $E = \text{tr}(e_i)$ for all $i = 1, \dots, n-1$. Then there exist $(c_n)_{n \in \mathbb{N}}$ in $\mathbb{C}(q, \zeta, u, z, E)$ such that, for all $n \in \mathbb{N}$,*

$$P(\hat{\alpha}) = c_n \Delta_S(\hat{\alpha}) \quad (\alpha \in B_n)$$

if and only if $P = \Delta_S$, that is, if and only if we are in one of the cases portrayed in the table of Theorem 5.

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